Comprehensive Examination in Algebra
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PART I: Do three of the following problems.

1. Let G be a finite group with identity element e, and suppose that α is an automorphism of G such that $\alpha(g) \neq g$ for $g \in G$ not equal to e. Suppose further that α^2 is the identity map on G. Prove that G is abelian. [Hint: First, prove that every element of G can be written in the form $x^{-1}\alpha(x)$. Second, apply α to such expressions and then determine a precise formulation for α.]

2. Let A be an additive (not necessarily finitely generated) abelian group. Recall if n is an integer that $nA = \{na : a \in A\}$ is a subgroup of A. (You do not have to prove that nA is a subgroup.) Now let p and q be relatively prime integers, and suppose that $(pq)A$ is the trivial (i.e., zero) subgroup. Prove that $A \cong pA \times qA$.

3. Let R be a commutative ring with multiplicative identity 1. Recall that an ideal P of R (with $1 \notin P$) is *prime* provided, for all $a, b \in R$, that $ab \in P$ implies $a \in P$ or $b \in P$.

   (a) Prove that every nonzero prime ideal in a PID must be a maximal ideal.

   (b) Prove the following statement: If $R[x]$ is a PID then R is a field.

4. Let R be an integral domain and $F \subset R$ be a subfield of R. Recall that $a \in R$ is *algebraic over* F if there exists a nonzero polynomial $p(x) \in R[x]$ such that $p(a) = 0$. Let $F[a]$ denote the smallest subring of R that contains both F and a. Prove that $F[a]$ is a field if and only if a is algebraic over F.

Part II: Do two of the following problems.

1. Let G be a finite group and let p_1,..., p_r be the distinct primes that divide $|G|$. Suppose that for any i, $1 \leq i \leq r$, G has a unique Sylow p_i-subgroup P_i.

   (a) Prove that $G \cong P_1 \times ... \times P_r$.

   (b) Let $H \leq G$ be a subgroup of G. Prove that H is isomorphic to the direct product of its Sylow p-subgroups.

2. Let R be a (commutative) integral domain (with nonzero multiplicative identity 1). We say that an element m of a left R-module M is *torsion* provided there exists a nonzero element $r \in R$ such that $r.m = 0$. A nonzero left R-module is *torsion* provided all of its elements are torsion, and a (possibly zero) module is *torsion free* provided none of its nonzero elements are torsion. We say that a nonzero left R-module is *uniform* if the intersection of any two of its nonzero left R-submodules is also nonzero. You may assume without proof that the set of torsion elements in a nonzero left R-module M forms a left R-submodule of M.

   (a) Let U be a uniform left R-module, and let V be a left R-submodule of U such that $(0) \subsetneq V \subsetneq U$. Prove that the left R-module U/V is torsion.

   (b) Prove that a nonzero uniform left R-module is either torsion or torsion free.

   (c) Give an example of a left $\mathbb{Z}$-module that is neither torsion nor torsion free.

3. (a) Let $m, n \in \mathbb{Z}$ be two integers such that neither m, n nor mn are perfect squares. Prove that $F = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ is a Galois extension of $\mathbb{Q}$ and that $Gal(F/\mathbb{Q}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

   (b) Conversely, let F be a Galois extension with $Gal(F/\mathbb{Q}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Prove that there exist integers $m, n \in \mathbb{Z}$ with neither m, n nor mn a perfect square such that $F = \mathbb{Q}(\sqrt{m}, \sqrt{n})$.