Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let C_∞ denote the infinite cyclic group and let G be an arbitrary group other than the trivial group $\langle 1 \rangle$. Show that the direct product $C_\infty \times G$ is not cyclic.

I.2 Let R be a ring (associative and with 1, but not necessarily commutative) and let I be a two-sided ideal of R. Assume that I is *nilpotent*, that is, $I^n = 0$ for some positive integer n.

(a) Show that an element $x \in R$ is invertible in R if and only if $x + I$ is invertible in R/I.

(b) Show that $\mathrm{GL}_n(R)$ maps *onto* $\mathrm{GL}_n(R/I)$ under the map that reduces all matrix entries in R modulo I.

(c) Give counterexamples to the statements in (a) and (b) when I is not nilpotent.

I.3 Let A be an $n \times n$-matrix over $\mathbb{R}$ such that some power of A is the identity matrix. Show that $\det(A) = (-1)^m$, where m is the multiplicity of -1 as root of the characteristic polynomial of A.

I.4 (a) Let $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be algebraic over $\mathbb{Q}$, not all 0, and let $z \in \mathbb{C}$ be a root of the polynomial $\alpha_n x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{C}[x]$. Prove that z is also algebraic over $\mathbb{Q}$.

(b) Let $z \in \mathbb{C}$ be transcendental over $\mathbb{Q}$. Show that $z - \sqrt{z}$ is also transcendental.

Part II. Do two of these problems.

II.1 Let $\mathbb{F}_q$ be the field with $q = p^a$ elements and let $G = \mathrm{GL}_3(\mathbb{F}_q)$ denote the group of invertible 3×3-matrices over $\mathbb{F}_q$. Furthermore, let $H \leq G$ be the subgroup consisting of all upper triangular matrices and $U \leq H$ the subgroup consisting of all upper triangular matrices with 1 on the diagonal.

(a) Prove that $|G| = (q^3-1)(q^3-q)(q^3-q^2)$.

(b) Show that U is a Sylow p-subgroup of G.

(c) Show that H is the normalizer of U in G and deduce a formula for the number of Sylow p-subgroups of G from this.

II.2 Let A be an $n\times n$-matrix with complex entries. Prove that

$$\det\big(\exp(A)\big) = \exp(\mathrm{tr}(A)),$$

where $\mathrm{tr}(A)$ denotes the trace of A and $\exp(A) := \lim_{N\to\infty}\sum_{k=0}^{N}\frac{1}{k!}A^k$. [You may assume that this limit exists for every A.]

II.3 (a) Give a careful definition of a finite Galois extension of fields.

(b) Give an example of field extensions $E \supset F$ and $F \supset K$, both of degree > 1 and both Galois, such that $E \supset K$ is not Galois. *Do not forget to prove that your extensions are Galois (resp. not Galois).*

(c) Prove that, for every finite group G, there exists a Galois field extension $E \supset F$ with

$$\mathrm{Gal}(E/F) \cong G.$$