January, 1998

Comprehensive Examination **Department of Mathematics**

ALGEBRA

PART I: Do three of the following problems.

1. Let G be a group.
   (a) Show that G is finite iff G has only finitely many distinct subgroups.
   (b) Show that G has exactly 3 distinct subgroups iff G is cyclic of order p^2 for some prime p.

2. Let A be a real $r \times r$-matrix satisfying $A^n = I$ for some $n > 0$. Prove: $\det A = (-1)^m$, where m is the multiplicity of -1 as root of the characteristic polynomial of A.

3. Let R be a ring and let N be an ideal of R. Assume that every element of $x \in N$ is nilpotent, that is, $x^t = 0$ for some t. Show that, under the canonical map $R \twoheadrightarrow R/N$, the group of units $\mathrm{U}(R)$ of R maps *onto* the group of units $\mathrm{U}(R/N)$ of R/N. (Recall that a *unit* of a ring R is an invertible element of R.)

4. (a) Let G be a finite group and let S and T be subsets of G (not necessarily distinct) with $|G| < |S| + |T|$. Show that $G = \{st \mid s \in S, t \in T\}$.
   (b) Conclude from part (i) that every element in a finite field is a sum of two squares.

PART II: Do two of the following problems.

1. Let G be a group of order pqr with distinct primes p, q, and r. Show that G is not simple.

2. Let $R = K[x, y]$ be the ring of polynomials in two variables x and y with coefficients in the field K, and let $f(x, y) \in R$.

   (a) Show that the principal ideal of R that is generated by $f(x, y)$ is prime if and only if the polynomial $f(x, y)$ is irreducible.

   (b) Show that the ideal of R that is generated by x and $f(x, y)$ is maximal if and only if the polynomial $f(0, y)$ is irreducible in $K[y]$.

3. Let F be the splitting field of $x^6 - 3$ over $\mathbb{Q}$.

   (a) Show that $[F : \mathbb{Q}] = 12$.

   (b) Let $G = \mathrm{Gal}(F/\mathbb{Q})$. Show that there exist a normal subgroup H of G of order 6 and a subgroup K of G of order 2 such that G is a semidirect product of H and K.

   (c) Determine whether the subgroup H of Part (b) is abelian.

Some alternative problems for consideration

1. Let S_n denote the symmetric group on n symbols and let $\sigma \in S_n$ be an n-cycle. Show that the centralizer of σ in S_n is exactly $\langle \sigma \rangle$.

2. Let A be a $n \times n$-matrix over a field F. Prove:

   (a) If A is nilpotent then $\text{trace}(A^m) = 0$ for all $m \geq 1$.

   (b) For $n = 2$ and F of characteristic $\neq 2$, prove the converse: If $\text{trace}(A) = 0 = \text{trace}(A^2)$ then $A^2 = 0$.

3. Let R be a ring, and let I and J be ideals of R. Suppose that $I \cap J = \{0\}$ and that R/I and R/J are commutative. Show that R is commutative.

4. Let $F \supseteq K$ be an algebraic extension of fields and let R be a subring of F with $R \supseteq K$. Show that R is a field.