Applied Math Exam

PART I. Do three of the following four problems.

1. Find the leading order asymptotic behavior for the integral

$$I(x) = \int_x^\infty e^{-t^4} dt$$

as $x \to \infty$.

2. A spring oscillating in a viscous medium has the equation of motion

$$m\frac{d^2y}{dt^2} = -\mu \left(\frac{dy}{dt}\right)^2 - ky$$

with initial conditions $y(0) = A$, $\dot{y}(0) = 0$. Scale to obtain a non-dimensional version and identify a nondimensional parameter ϵ. Provide an intuitive interpretation for ϵ

3. Consider the problem

$$\nabla^2 u = C, \quad \frac{\partial u}{\partial r} = 0 \text{ on } r = 1,$$

on the domain $r^2 = x^2 + y^2 < 1$. For which values of C does a solution exist?

4. Find the Green's function for the problem

$$u'' - u = f(x). \quad 0 < x < 1,$$

with boundary conditions $u(0) = u(1) = 0$.

PART II. Do two of the following three problems.

1. Suppose that the dynamical system

$$\frac{d}{dt}\mathbf{u} = \mathbf{f}(\mathbf{u})$$

where

$$\mathbf{f}(\mathbf{u}) = \nabla F(\mathbf{u})$$

(here the gradient is taken with respect to $\mathbf{u}$). Show that the dynamics cannot include a limit cycle.

2. Find an approximate solution to

$$\epsilon y'' + \epsilon y' - y^2 = -1 - x^2, \quad y(0) = 2, \quad y(1) = 2,$$

uniformly valid to $\mathrm{O}(\epsilon)$.

3. Consider the equation

$$u_{xx} + u_{yy} = 0$$

in the upper half plane $y > 0$, and

(a) the Dirichlet data $u(x, 0) = f(x)$,

(b) the Neumann data $u_y(x, 0) = g(x)$,

where f and g are 2π periodic in x. Assume u is bounded at infinity and also is 2π periodic in x. Find the map

$$L\hat{f}(\mathbf{k}) = \hat{g}(\mathbf{k})$$

of the Dirichlet-to-Neumann transform, where $\hat{f}$, $\hat{g}$ are the Fouriere transforms of f, g, respectively.