PH.D. COMPREHENSIVE EXAMINATION
COMPLEX ANALYSIS SECTION

August 1994

Part I. Do three (3) of these problems.

I.1. Suppose that D is a region and that $f : D \to \mathbb{C}$ is a one-to-one analytic function. Show that $f'(x) \neq 0$ for every $z \in D$.

I.2. Let f be analytic in

$$D = \{z : 0 < |z| < 1\}$$

except for a sequence of poles $\{a_n\}$ in D with $a_n \to 0$. Show that for any $z \in \mathbb{C}$ there is a sequence $\{b_n\}$ in D with $\lim b_n = 0$ and

$$\lim f(b_n) = z.$$

I.3. Suppose that $f(z)$ is entire and $\mathrm{Im} f(z) \neq 0$ whenever $|z| \neq 1$. Prove that $f(z)$ is constant.

I.4. Evaluate the following integral, justifying all of your steps.

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + x + 1} dx$$

Part II. Do two (2) of these problems.

II.1. Suppose that f is analytic on the unit disc and $|f(z)| \leq 1$ when $|z| < 1$. Prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}.$$

II.2. Prove that there is no one-to one analytic function which maps the punctured disc $D = \{z : 0 < |z| < 1\}$ onto the annulus $\Omega = \{z : 2 < |z| < 4\}$.

II.3. (a) Construct an entire function with simpe zeros at the points $0, 1, 2^2, 3^3, 4^2, \ldots,$ *and no other zeros.*

(b) Construct an entire function with simpe zeros at the points $0, 1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots,$ *and no other zeros.*