Ph.D. Comprehensive Examination
Complex Analysis

August 2012

Part I. Do three of these problems.

I.1 Compute

$$\int_0^\infty \frac{\sin x}{x(1+x^2)}\, dx.$$

Prove all claims.

I.2 Let $f(z)$ be analytic on the open unit disc D. Prove that

(i) $f(z)$ has a primitive in D.

(ii) If $f(z) \neq 0$ for any $z \in D$, then there exists a function $g(z)$ analytic in D such that $f(z) = e^{g(z)}$.

(iii) If $f(z) \neq 0$ for any $z \in D$, then for any integer n there exists a function $h(z)$ analytic in D such that $f(z) = (h(z))^n$.

I.3 (i) Let $f(z)$ be an entire function. Suppose that $\text{Re}(f(z)) > 0$ for any $z \in \mathbb{C}$. Show that $f(z)$ is a constant function.

(ii) Let $u_1(x, y)$ and $u_2(x, y)$ be two harmonic functions in $\mathbb{C}$. Prove that either $u_1(x, y) - u_2(x, y)$ is a constant function or there exists $x + iy \in \mathbb{C}$ such that $u_1(x, y) = u_2(x, y)$.

I.4 Let f and g be entire functions one of which is a polynomial. Suppose that $f \not\equiv 0$ and that there is $c > 0$ such that $|f(z)| \leq c|g(z)|$ for all z. Show that the other function is also a polynomial.

Part II. Do two of these problems.

II.1 Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose $f : \overline{D} \to \mathbb{C}$ is continuous, nonconstant, holomorphic in D, and $f(\partial D) \subset \partial D$.

(i) Show that for any $z \in D$, $f(z) \in D$.

(ii) Let $w \in D$ and let $n(w)$ denote the number of zeros of the function $f(z) - w$ in D, counted with multiplicities. Show that $n(w)$ is finite and is independent of w.

(iii) Show that $f : D \to D$ is onto.

II.2 Let G be a region in $\mathbb{C}$, $a_1, a_2 \in G$, and

$$\mathcal{F} = \{f : G \to \mathbb{C} : f \text{ is holomorphic and } |f(z)| \leq 1\}.$$

Define $\Phi : \mathcal{F} \to \mathbb{C}$ by $\Phi(f) = |f(a_1)| + |f'(a_2)|$. Show that Φ has a maximum: there is $f_0 \in \mathcal{F}$ such that $\Phi(f_0) = \sup\{\Phi(f) : f \in \mathcal{F}\}$.

II.3 An entire function $f(z)$ is called a function of finite order if there exists a $\lambda > 0$ such that for all z with $|z|$ sufficiently large $|f(z)| < e^{|z|^\lambda}$. Prove that sums, products, derivatives and primitives of entire functions of finite order are also entire functions of finite order.

Typo corrected Aug 30