Ph.D. Comprehensive Examination
Complex Analysis

August 2013

Part I. Do three of these problems.

I.1 Show that there is no analytic function f on the annulus $\{z : 1 < |z| < 2\}$ such that $e^{f(z)} = z$ for every z in the annulus.

I.2 For $-1 < a < 3$ compute

$$\int_0^\infty \frac{x^a}{(1+x^2)^2}\, dx.$$

Prove all claims.

I.3 Find a conformal map from $G = \{z : |z| < 1, \Re(z) > 0\}$ onto $D = \{z : |z| < 1\}$.

I.4 Find the number of zeros of $z^7 + 4z^4 + z^3 + 1$ in the annulus $\{z : 1 < |z| < 2\}$.

Part II. Do two of these problems.

II.1 Let G be a bounded simply connected domain and $a \in G$. Suppose $f : G \to G$ is an analytic function such that $f(a) = a$. Prove that f is one-to-one and onto if and only if $|f'(a)| = 1$.

II.2 Let G be an open connected subset of $\mathbb{C}$ and let $\{f_n\}_n$ be a sequence of analytic functions on G that are uniformly bounded. Let S be the set of points in G where the sequence is convergent. If S has a limit point in G, prove that $\{f_n\}_n$ converges uniformly on compact subsets of G.

II.3 Find a function f, analytic on $|z| < 1$ and such that $f(z) = 0$ if and only if $z = 1 - \frac{1}{\sqrt{k}}$, $k = 1, 2, \ldots$. You may leave your answer as an infinite product. Justify all claims.