Ph.D. Comprehensive Examination
Complex Analysis

August 2015

Part I. Do three of these problems.

I.1 Let $D \subset \mathbb{C}$ be an open nonempty connected set and consider $h : D \longrightarrow \mathbb{C}$ a holomorphic function satisfying

$$\big(h(z)\big)^2 = \overline{h(z)} \quad \forall\, z \in D.$$

(a) Show that the function h is constant on D.

(b) Find all possible values for the function h satisfying the above property.

I.2 Consider the function $u : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$ given by $u(x,y) := \ln(x^2 + y^2)$ for each $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

(a) Show that u is harmonic in $\mathbb{R}^2 \setminus \{(0,0)\}$.

(b) Show that there is no holomorphic function $f : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$ such that

$$u(x,y) = \mathrm{Re} f(x+iy) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

I.3 Consider the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ given by $f(z) := e^{z^2} - 4z^2$ for each $z \in \mathbb{C}$. Determine the number of zeroes (counted with multiplicities) of f in the unit disk.

I.4 Let f be an entire function such that $|f(z)| \geq |z|^k$ with $k \geq 1$ when $|z|$ is large enough. Show that f is a polynomial of degree at least k.

Part II. Do two of these problems.

II.1 Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $K \subset D$ a compact subset. Suppose that $f : D\backslash K \to \mathbb{C}$ is holomorphic and that there is a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that $p_n \to f$ as $n \to \infty$ uniformly on compact subsets of $D\backslash K$. Show that there exists a holomorphic function $g : D \to \mathbb{C}$ such that $f = g$ on $D\backslash K$.

II.2 Let $G \subset \mathbb{C}$ be an open set, let $\{h_n\}_{n=1}^{\infty}$ be a uniformly bounded sequence of harmonic functions $h_n : G \to \mathbb{R}$, for each $n \in \mathbb{N}$, which is pointwise increasing.

a) Show that the pointwise defined limit function $h : G \to \mathbb{R}$ given by $h(z) = \lim_{n\to\infty} h_n(z)$ for each $z \in G$ is continuous.

b) Show that in fact h is harmonic in G.

Hint. The mean value theorem for harmonic functions and Harnack's inequality (if $u \geq 0$ is harmonic in a neighborhood of the closure of the disc $B(a, R)$, then $(R - r)u(a)/(R + r) \leq u(a + re^{i\theta}) \leq (R + r)u(a)/(R - r)$ for $0 \leq r < R$ and $\theta \in \mathbb{R}$) may be of use.

II.3 Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and suppose that the function f is defined and continuous in $A = D \cap \{z : \mathrm{Im}(z) \geq 0\}$, holomorphic in $D \cap \{z : \mathrm{Im}(z) > 0\}$, real-valued on $I = (-1, 1)$. Suppose there is a sequence $\{x_n\}_{n=1}^{\infty} \subset I$ of distinct points tending to 0 with $f(x_n) - x_n^2 = 0$. Show that $f(z) = z^2$ in A.

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem. For each $z_o \in \mathbb{C}$ and $r > 0$, the open ball in $\mathbb{C}$ with center z_o and radius r is denoted by $B(z_o, r)$.