PH.D. COMPREHENSIVE EXAMINATION
COMPLEX ANALYSIS SECTION

January 1995

Part I. Do three (3) of these problems.

I.1. Suppose f is analytic in a domain D, and let a be a point in D. Let $\{z_n\}$, $\{w_n\}$ be two sequences of points in D such that $z_n \neq w_n$ for all n, and $\lim_{n\to\infty} z_n = a$, $\lim_{n\to\infty} w_n = a$. Show that

$$\lim_{n\to\infty} \frac{f(w_n) - f(z_n)}{w_n - z_n} = f'(a).$$

I.2. Evaluate $\int_C e^z(z+1)^{-4}dz$ where C is the imaginary axis from $-i\infty$ to $+i\infty$.

I.3. If f is entire and $f(-z) = f(z)$ for all z, then there is an entire function g satisfying $f(z) = g(z^2)$ for all z.

I.4. Let $D = \{z : 0 < \arg z < 3\pi/2\}$. Find a function u which is continuous on $\bar{D} \setminus \{0\}$, harmonic in D, and satisfying $u(x,0) = 1$ for $x > 0$ and $u(0,y) = 0$ for $y < 0$.

Part II. Do two (2) of these problems.

II.1. Let $H = \{z : \text{Im}(z) \geq 0\}$. Suppose $F : H \to H$ is analytic and $a \in H$. Prove that

$$|F'(a)| \leq \frac{\text{Im}F(a)}{\text{Im}a}.$$

II.2. Suppose f is a polynomial of degree $n \geq 1$ and satisfies $|f(z)| \leq 1$ on the unit disc. Show that $|f(z)| \leq |z|^n$ if $|z| \geq 1$.

II.3. Suppose f is holomorphic in the unit disk D and continuous on $\bar{D}$ and thus

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad z \in D.$$

If f has exactly m zeroes in D, show that

$$\min\{|f(z)| : |z| = 1\} \leq |c_0| + |c_1| + \cdots + |c_m|.$$