PH.D. COMPREHENSIVE EXAMINATION
COMPLEX ANALYSIS SECTION

January 1996

Part I. Do three (3) of these problems.

I.1. Let $\mathbb{R}^- = \{x \in \mathbb{R} : x \leq 0\}$. Suppose $f(z)$ is analytic on $\mathbb{C} \setminus \mathbb{R}^-$ and $f(x) = x^x$ for $x \in \mathbb{R}$, $x > 0$. Find $f(i)$ and $f(-i)$.

I.2. Let $f(z)$ be an analytic function on an open connected subset $G \subset \mathbb{C}$. Suppose that $f(z)$ maps G onto a subset of a straight line. Show that $f(z)$ is a constant.

I.3. Find a conformal mapping from the region $\{z \in \mathbb{C} : |z-1| > 1 \text{ and } |z+1| > 1\}$ onto the punctured disc $D = \{z \in \mathbb{C} : |z| < 1\} \backslash 0$. Hint: Apply $T(z) = \frac{1}{z}$ first.

I.4. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx$ using residues.

Part II. Do two (2) of these problems.

II.1. Let G_1 and G_2 be two bounded simply connected regions, and let $z_0 \in G_1$ and $w_0 \in G_2$. Show that there exists a bijective analytic mapping $f(z)$ from G_1 to G_2 such that $f(z_0) = w_0$.

II.2. Let $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$, $Re(z) > 0$. Show that $\Gamma(z+1) = z\Gamma(z)$, use this formula to obtain a meromorphic continuation of $\Gamma(z)$ to the entire complex plane, and find the poles of $\Gamma(z)$ on $\mathbb{C}$, their orders and residues.

II.3. i) Let $u(x, y)$ be a harmonic function on the disc $D = \{z : |z - z_0| < R\}$. Show that for any $r < R$, $u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$.

ii) Let $u(x, y)$ be a harmonic function on a bounded region G that is continuous on the closure $\overline{G}$ of G. Show that $u(x, y)$ achieves its maximum and minimum values on the boundary of G.