**Ph.D. Comprehensive Examination**
**Complex Analysis Section**
**January 2002**

**Part I.** Do three (3) of these problems.

**I.1.** Let

$$D = \{z = x + iy : \; x^2 + y^2 < 1 \text{ and } x^2 - x + y^2 > 0\}.$$

Find a conformal mapping from $D$ to the open unit disc. Hint: start with a Möbius transformation that takes 1 to $\infty$.

**I.2.** Show that for any $0 < a < 1$

$$\int_0^\infty \frac{x^a}{x(1+x)} dx = \frac{\pi}{\sin(\pi a)}.$$

Be sure to justify your answer carefully.

**I.3.** Suppose $f$ is meromorphic on $\mathbb{C}$ and there are positive numbers $C$, $k$ and $R$ such that $|f(z)| < C|z|^k$ if $|z| > R$. Show that $f$ is rational.

**I.4.** Let $D = \{z : 0 < \arg z < \pi/2\}$. Find a function $u$ which is continuous on $\overline{D} \setminus \{0\}$, harmonic in $D$, and satisfies $u(x, 0) = 1$ for $x > 0$ and $u(0, y) = 0$ for $y > 0$.

**Part II.** Do two (2) of these problems.

**II.1.** Suppose $f(z)$ is analytic on the punctured unit disc $\{z : \; 0 < |z| < 1\}$ except for a sequence of poles $z_n$ that converges to 0. Show that for any $\varepsilon > 0$, $f(\{z : \; 0 < |z| < \varepsilon\})$ is everywhere dense in $\mathbb{C}$.

**II.2.** Let $f(z)$ be an entire function. Suppose there exist $M > 0$ and a sequence $\{R_n\}$ of positive numbers tending to $\infty$ with $f(z) \neq 0$ on $|z| = R_n$, such that

$$\oint_{|z|=R_n} \left| \frac{f'(z)}{f(z)} \right| |dz| < M \quad \text{ for all } n.$$

Show that $f(z)$ is a polynomial.

**II.3.** Suppose $f$ is holomorphic in the unit disk $D$ and continuous on $\bar{D}$ and thus

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in D.$$

Show that if $f$ has exactly $m$ zeroes counted with multiplicity in $D$, then

$$\min\{|f(z)| : |z| = 1\} \leq |c_0| + |c_1| + \cdots + |c_m|.$$