PH.D. COMPREHENSIVE EXAMINATION
COMPLEX ANALYSIS SECTION

January, 2006

Part I. Do three (3) of these problems.

I.1. Find a conformal one-to-one map of the half disc $\{z : |z| < 2, \operatorname{Re} z < 0\}$ onto the unit disc $U = \{z : |z| < 1\}$.

I.2. Let $f(z) = u(z) + i\, v(z)$ be analytic in the unit disc $U = \{z : |z| < 1\}$, u and v real. Show that if $u(0)^2 = v(0)^2$, then

$$\int_0^{2\pi} u(re^{i\theta})^2\, d\theta = \int_0^{2\pi} v(re^{i\theta})^2\, d\theta \quad \text{for} \quad 0 < r < 1.$$

I.3. Let $f : [a,b] \longrightarrow \mathbb{C}$ be continuous $(a < b)$. Let g be defined by

$$g(z) = \int_a^b \frac{f(t)}{z - t}\, dt.$$

Prove that g is analytic in $\mathbb{C} / [a,b]$.

I.4. Suppose f is meromorphic on $\mathbb{C}$ and there exist $K, k, R > 0$ such that $|f(z)| < K|z|^k$ if $|z| > R$. Prove that f is a rational function.

Part II. Do two (2) of these problems.

II.1.

(a) Let U = the unit disc. Suppose $f(z)$ is analytic in U, continuous on $\overline{U}$ and real-valued on ∂U. Prove that f is constant.

(b) If $f(z)$ is as in (a), except that $f(z)$ is assumed real-valued only on the arc $\gamma = \{e^{i\theta} : 0 < \theta < \frac{\pi}{10}\}$, what can we conclude? Explain.

II.2. Let f be analytic in the region $|z| > 1$. Prove that if f is real-valued on $(1, \infty)$, then it is also real-valued on $(-\infty, -1)$.

II.3. Let $\{f_n\}$ be a sequence of analytic functions on a region G that converges uniformly on G. If K is a compact subset of G, prove that the sequence of derivatives $\{f_n'\}$ converges uniformly on K.