PDEs Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Solve
$$\begin{cases} \dfrac{\partial u}{\partial x} + x\,\dfrac{\partial u}{\partial y} = u \\ u(1,y) = h(y) \end{cases}$$
where $h \in C^1(\mathbb{R})$.

2. Let $k \in \mathbb{R}$ and let
$$\Gamma(x) = \frac{e^{ik|x|}}{|x|} \qquad x \in \mathbb{R}^3, x \neq 0.$$
Prove that Γ satisfies the Helmholtz equation $\Delta\Gamma + k^2\,\Gamma = 0$ for $x \neq 0$.

3. Let $f \in L^1(\mathbb{R}^n)$ and its Fourier transform $\hat{f}(x) = \int_{\mathbb{R}^n} f(y)\,e^{-2\pi i\,x\cdot y}\,dy$. If $g(x) = |x|\,f(x)$ belongs to $L^1(\mathbb{R}^n)$, then prove that $\hat{f}$ satisfies the Lipschitz estimate
$$\left|\hat{f}(x) - \hat{f}(y)\right| \leq 2\,\pi\,\|g\|_1\,|x-y| \qquad \forall x, y \in \mathbb{R}^n.$$

4. Let $F, G : \mathbb{R} \to \mathbb{R}$ be continuous and let $w(\xi,\eta) = F(\xi) + G(\eta)$. Prove that w is a generalized solution to the equation $w_{\xi\eta} = 0$, that is,
$$\int_{\mathbb{R}^2} w(\xi,\eta)\,\phi_{\xi\eta}(\xi,\eta)\,d\xi d\eta = 0 \qquad \forall \phi \in C_0^2(\mathbb{R}^2).$$
Conclude that $u(x,t) = F(x+ct) + G(x-ct)$ is a generalized solution to the wave equation $\Box u = u_{tt} - c^2 u_{xx} = 0$, that is, $\int_{\mathbb{R}^2} u(x,t)\,\Box\phi(x,t)\,dxdt = 0$ for all $\phi \in C_0^2(\mathbb{R}^2)$.

Part II. (Do 2 problems)

1. Let $u(x,t)$ be a C^2 bounded solution of
$$u_t(x,t) - u_{xx}(x,t) = 0,\ x \in \mathbb{R},\ t > 0,\ u(x,0) = f(x)$$
where $f \in C(\mathbb{R})$ satisfies:
$$\lim_{x\to+\infty} f(x) = A,\ \lim_{x\to-\infty} f(x) = B$$
for some constants A and B. Show that $\lim_{t\to\infty} u(x,t) = \dfrac{A+B}{2}$, for each $x \in \mathbb{R}$.
Hint: Use an integral representation for the solution u. Justify why the representation is valid.

2. If $f \in W^{1,2}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ connected and $Df = 0$, then prove that f is constant in Ω.

3. Let B be a ball in $\mathbb{R}^n$, $f \in C(\partial B)$, and the boundary value problem
$$\begin{cases} \Delta u &= 1 \text{ in } B \\ \dfrac{\partial u}{\partial \nu} &= f \text{ on } \partial B. \end{cases}$$
Prove that
   1. if $u_1, u_2 \in C^2(\bar{B})$ solve the boundary value problem, then $u_1 - u_2$ is constant in B;
   2. if there is a solution $u \in C^2(\bar{B})$ to the boundary value problem, then $\int_{\partial B} f(x)\,d\sigma(x) = |B|$.