PDEs Ph.D. Qualifying Exam
Temple University
January, 2015

Part I. (Do 3 problems)

1. Solve the initial value problem

$$x^2 u_x + xy\, u_y = u^2 \qquad u(y^2, y) = 1.$$

2. Compute the Fourier transform of $u(x) = x\, e^{-x^2}$.

3. Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^1 boundary and suppose u_1 and u_2 are two functions in $C^2(\overline{\Omega})$ that are solutions of

$$\Delta u_1 = \lambda_1 u_1,\ \Delta u_2 = \lambda_2 u_2 \text{ in } \Omega, u_1 = u_2 = 0 \text{ on } \partial\Omega,$$

where λ_1 and λ_2 are two constants, $\lambda_1 \neq \lambda_2$. Show that $\int_\Omega u_1(x) u_2(x)\, dx = 0$.

4. Let u be harmonic in $\mathbb{R}^n$. Prove that
   (a) $\Delta(u^2) \geq 0$ in $\mathbb{R}^n$;
   (b) if $\int_{\mathbb{R}^n} u(x)^2\, dx < +\infty$, then $u \equiv 0$.

Part II. (Do 2 problems)

1. Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $c(x)$ continuous and strictly positive in $\bar{\Omega}$, and $\alpha(x) \geq 0$ continuous in $\partial\Omega$. Suppose $u(x,t)$ is a smooth solution to

$$\begin{aligned} u_{tt} - c(x)^2\, \Delta u &= 0 \quad \text{in } \Omega \times [0, T] \\ u_t - \alpha(x)\partial_\nu u &= 0 \quad \text{in } \partial\Omega \times [0, T]. \end{aligned}$$

Prove that the energy

$$E(t) = \frac{1}{2}\int_\Omega \left(\frac{1}{c(x)^2} u_t^2 + |Du|^2 \right) dx$$

satisfies $\dfrac{dE}{dt} \geq 0$ for $0 \leq t \leq T$. Here $\partial_\nu u$ denotes the outer normal derivative of u.

2. Suppose $\Omega \subset \mathbb{R}^n$ is a connected domain and $u \in W^{1,p}(\Omega)$, for some $1 \leq p < \infty$, with weak derivatives $\dfrac{\partial u}{\partial x_j} = 0$ for $1 \leq j \leq n$. Prove that u is constant in Ω.

3. Let $u(x,t)$ be a solution to the heat equation $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, +\infty)$. Suppose that $\sup_{|x|<R} |u(x,t) - A(x)| \to 0$ as $t \to +\infty$ for some function $A(x)$. Prove that A is harmonic in $|x| < R$.

HINT: prove that A is weakly harmonic in $|x| < R$, that is, $\int_{\mathbb{R}^n} A(x)\, \Delta\phi(x)\, dx = 0$ for all $\phi \in C_0^\infty(|x| < R)$. Using the equation and the divergence theorem show first that $\int_{t_1}^{t_2} \int_{\mathbb{R}^n} u(x,t)\, \Delta\phi(x)\, dxdt = \int_{\mathbb{R}^n} \phi(x)\, (u(x,t_2) - u(x,t_1))\, dx$. Next write $\int_{t_1}^{t_2} \int_{\mathbb{R}^n} A(x)\, \Delta\phi(x)\, dxdt = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} (A(x) - u(x,t))\, \Delta\phi(x)\, dxdt + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} u(x,t)\, \Delta\phi(x)\, dxdt$.