PDEs Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Solve the damped Burgers' equation

$$\begin{aligned} u\,u_x + u_y &= -u, \text{ for } x \in \mathbb{R}, y > 0, \\ u(x,0) &= x. \end{aligned}$$

2. Let $u(x,t)$ solve the heat equation

$$\begin{aligned} u_t &= \Delta u, \text{ for } x \in \mathbb{R}^n, t > 0, \\ u &= f \text{ for } t = 0, \end{aligned}$$

with the usual growth condition to guarantee uniqueness in place. Show that

$$\|u(\cdot,t)\|_{L^p} \le \|f\|_{L^p}$$

for any $p \ge 1$ and all $t > 0$.

3. Show that if $f \in H^1(\Omega)$ for $\Omega \subset \mathbb{R}^1$, then f is Hölder continuous with exponent $1/2$. Show that if $\mathbb{R}^1$ is replaced by $\mathbb{R}^n$, $n > 1$, then f need not even be continuous.

4. Let $f \in L^1(\mathbb{R}^n)$ and its Fourier transform $\hat{f}(x) = \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i\, x\cdot y}\, dy$. If $g(x) = |x|\, f(x)$ belongs to $L^1(\mathbb{R}^n)$, then prove that $\hat{f}$ satisfies the Lipschitz estimate

$$\left|\hat{f}(x) - \hat{f}(y)\right| \le 2\,\pi\, \|g\|_1\, |x-y| \qquad \forall x, y \in \mathbb{R}^n.$$

Part II. (Do 2 problems)

1. Consider $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solution to the boundary value problem

$$\Delta u = c\,u - |\nabla u|^2, \text{ in } \Omega,$$
$$u = 0, \text{ on } \partial\Omega,$$

   where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Show that if $c(x) > 0$ for all $x \in \Omega$, then $u \equiv 0$ in Ω.

2. Let $u = u(x,t) \in C^2([0,1] \times [0,\infty))$ be a solution to

$$u_{tt} - u_{xx} = -\frac{u}{1+u^2}, \text{ for } 0 < x < 1, t > 0,$$
$$u_t(1,t)\,u_x(1,t) - u_t(0,t)\,u_x(0,t) = 0, \text{ for } t > 0.$$

   (a) Find a function ϕ so that the energy

$$E(t) = \int_0^1 \left(u_t^2 + u_x^2 + \phi(u)\right) dx$$

   is constant in time.

   (b) In addition, if $u(0,t) = 0$ for all $t > 0$, then conclude that there is a constant $c > 0$ so that $|u(x,t)| \leq c\,x^{1/2}$ for all $x \in [0,1]$ and $t > 0$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let u solve the eigenvalue problem

$$-\Delta u = -u^3 + \lambda\,u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

   Here $u \not\equiv 0$ and $\lambda \in \mathbb{R}$.

   Prove the following

   (a) $\lambda = \dfrac{\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^4\,dx}{\int_\Omega u^2\,dx}$;

   (b) there cannot exist a sequence of eigen-pairs (u_k, λ_k) such that $\lambda_k \to 0$.