Ph.D. Comprehensive Examination
Real Analysis

Spring 2017

Part I. Do three of these problems.

I.1. Give examples of the following (and prove your assertions):

i) A continuous function $f : \mathbb{R} \to \mathbb{R}$ that is bounded but not uniformly continuous.

ii) A uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ that is not bounded.

iii) A function $f : [0,1] \to \mathbb{R}$ that is continuous but not Lipschitz continuous.

I.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function with f and $f' \in L^1(\mathbb{R})$ (with respect to Lebesgue measure). Show:

i) Both limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist.

ii) $\lim_{\lambda\to\infty} \int_{\mathbb{R}} f(x)\cos(\lambda x)\,dx = 0.$

Hint for part *ii*): $\cos t = \dfrac{d\sin t}{dt}$.

I.3. (a) Find the limit and justify your answer:

$$\lim_{n\to\infty} \int_0^1 \frac{n^{1/2}x}{1+nx^2}\,dx.$$

(b) Find the limit and justify your answer:

$$\lim_{n\to\infty} \int_0^n \big(1-\frac{x}{n}\big)^n \cos\big(\frac{x}{n}\big)\,dx.$$

I.4. Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \subset X$ measurable with finite measure. Suppose $f : E \to \mathbb{R}$ is measurable. Let E_k be measurable subsets of E. Suppose $\int_E |f|\,d\mu < \infty$ and $\lim_{k\to\infty} \mu(E_k) = 0$. Show that $\lim_{k\to\infty} \int_{E_k} |f|\,d\mu = 0$.

Part II on next page

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part II. Do two of these problems.

II.1. Let $I = [-1,1] \subset \mathbb{R}$. Recall that $f : I \to \mathbb{R}$ is said to be absolutely continuous if it has the property that

> for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every disjoint family $\{[a_\ell, b_\ell]\}_{\ell=1}^N$ of subintervals of I,
>
> $$\sum_{\ell=1}^N |b_\ell - a_\ell| < \delta \implies \sum_{\ell=1}^N |f(b_\ell) - f(a_\ell)| < \varepsilon.$$

Let A be the set of absolutely continuous functions on I. Show:

i) A is a subalgebra of the algebra $C(I,\mathbb{R})$ of continuous real valued functions $I \to \mathbb{R}$.

ii) If $f : I \to \mathbb{R}$ is Lipschitz continuous on I, then it is absolutely continuous.

iii) A is dense in $C(I,\mathbb{R})$.

II.2. Let $\{a_n\}_{n=0}^\infty$ be a sequence of real numbers converging to 0 as $n \to \infty$. Define $A, A_N : \ell^2 \to \ell^2$ by

$$Ax = \{a_n x_n\}_{n=0}^\infty,\ A_N x = \{a_n x_n\}_{n=N}^\infty \text{ for } x = \{x_n\}_{n=0}^\infty \in \ell^2.$$

i) Show that for every $\mu \in \mathbb{N}$ there is $N_\mu \in \mathbb{N}$ such that

$$N \geq N_\mu \implies \text{ for all } \{x_n\}_{n=0}^\infty \in \ell^2 : \|\{a_n x_n\}_{n=N}^\infty\|_{\ell^2} \leq \frac{1}{\mu}\|\{x_n\}_{n=0}^\infty\|_{\ell^2}$$

ii) Show that every bounded sequence $\{x^{(j)}\}_{j=1}^\infty$ in ℓ^2 has a subsequence $\{x^{(j_k)}\}_{k=1}^\infty$ such that $Ax^{(j_k)}$ converges in ℓ^2.

Hint: Bounded sequences in finite dimensional spaces have convergent subsequences.

II.3. Suppose f and $xf(x)$ are both integrable on $\mathbb{R}$. Let $F(k) = \int_{-\infty}^\infty e^{ikx} f(x)dx,\ k \in \mathbb{R}$. Show that F is continuously differentiable and $F'(k) = i\int_{-\infty}^\infty e^{ikx} x f(x)dx$.