Abstract. This paper proves that every finite volume hyperbolic 3–manifold \( M \) contains a ubiquitous collection of closed, immersed, quasi-Fuchsian surfaces. These surfaces are ubiquitous in the sense that their preimages in the universal cover separate any pair of disjoint, non-asymptotic geodesic planes. The proof relies in a crucial way on the corresponding theorem of Kahn and Markovic for closed 3–manifolds. As a corollary of this result and a companion statement about surfaces with cusps, we recover Wise’s theorem that the fundamental group of \( M \) acts freely and cocompactly on a CAT(0) cube complex.

1. Introduction

A collection of immersed surfaces in a hyperbolic 3–manifold \( M = \mathbb{H}^3 / \Gamma \) is called ubiquitous if, for any pair of hyperbolic planes \( \Pi, \Pi' \subset \mathbb{H}^3 \) whose distance is \( d(\Pi, \Pi') > 0 \), there is some surface \( S \) in the collection with an embedded preimage \( \tilde{S} \subset \mathbb{H}^3 \) that separates \( \Pi \) from \( \Pi' \). The main result of this paper is the following.

Theorem 1.1. Let \( M = \mathbb{H}^3 / \Gamma \) be a complete, finite volume hyperbolic 3–manifold. Then the set of closed immersed quasi-Fuchsian surfaces in \( M \) is ubiquitous.

We refer the reader to Section 2.2 for the definition of a quasi-Fuchsian surface (abbreviated QF). Informally, a quasi-Fuchsian subgroup of isometries of \( \mathbb{H}^3 \) preserves a small deformation of a totally geodesic hyperbolic plane.

Theorem 1.1 resolves a question posed by Agol [15, Problem 3.5]. Kahn and Markovic proved exactly the same statement under the additional hypothesis that \( M \) is closed [24]. Their theorem is a crucial ingredient in our proof of Theorem 1.1. Very recently, Kahn and Wright have outlined an independent proof of Theorem 1.1 by modifying the dynamical methods of Kahn and Markovic [24], including the good pants homology [25].

A slope on a torus \( T \) is an isotopy class of essential simple closed curves on \( T \), or equivalently a primitive homology class (up to sign) in \( H_1(T) \). When \( M \) is a cusped hyperbolic 3–manifold — that is, non-compact with finite volume — it follows from the work of Culler and Shalen [14] that \( M \) contains at least two embedded QF surfaces with cusps, and furthermore that the boundaries of these surfaces have distinct slopes on each cusp torus of \( M \). Masters and Zhang [28, 29], as well as Baker and Cooper [6], found ways to glue together covers of these cusped QF surfaces to produce a closed, immersed QF surface. However, it is not clear whether these constructions can produce a ubiquitous collection of QF surfaces.

If an embedded essential surface \( S \subset M \) has all components of \( \partial S \) homotopic to the same slope \( \alpha \subset M \), we say that \( \alpha \) is an embedded boundary slope. An immersed boundary slope in \( M \) is a slope \( \alpha \) in a cusp torus of \( M \), for which there is an integer \( m > 0 \) and an essential immersed surface \( S \)...
whose boundary maps to loops each homotopic to ±m·α. Such a surface S is said to have immersed slope α. We prove the following.

**Theorem 1.2.** Let \( M = \mathbb{H}^3/\Gamma \) be a cusped hyperbolic 3–manifold, and let \( \alpha \) be a slope on \( \partial M \). Then the set of cusped quasi-Fuchsian surfaces immersed in M with immersed slope \( \alpha \) is ubiquitous.

Hatcher showed that a compact manifold bounded by a torus has only finitely many embedded boundary slopes [19]. Hass, Rubinstein, and Wang [18] (refined by Zhang [40]) showed that there are only finitely many immersed boundary slopes whose surfaces have bounded Euler characteristic. Baker [2] gave the first example of a hyperbolic manifold with infinitely many immersed boundary slopes, while Baker and Cooper [3] showed that all slopes of even numerator in the figure-eight knot complement are virtual boundary slopes. Oertel [30] gave a manifold with one cusp so that all slopes are immersed boundary slopes. Maher [27] gave many families, including all 2–bridge knots, for which every slope is an immersed boundary slope. Subsequently, Baker and Cooper [4] showed that all slopes of one-cusped manifolds are immersed boundary slopes. However, the surfaces constructed in those papers are not quasi-Fuchsian, because they contain annuli parallel to the boundary. By contrast, Theorem 1.2 produces a ubiquitous collection of QF surfaces realizing every slope as an immersed boundary slope.

Theorems 1.1 and 1.2, combined with results of Bergeron and Wise [7] and Hruska and Wise [22], have the following immediate consequence. (See Section 8 for definitions related to cube complexes.)

**Corollary 1.3.** Let \( M \) be a complete, finite volume hyperbolic 3–manifold. Then \( \pi_1(M) \) acts freely and cocompactly on a CAT(0) cube complex dual to finitely many immersed quasi-Fuchsian surfaces.

Corollary 1.3 is not new. When \( M \) is closed, the result is due to Bergeron and Wise [7, Theorem 1.5], using the closed case of Theorem 1.1 proved by Kahn and Markovic [24]. When \( M \) is cusped, the result is due to Wise [38, Theorem 16.28], and occurs in the last step of his construction of a virtual quasiconvex hierarchy for \( \pi_1(M) \). Despite the lack of novelty, our proof of the cusped case using Theorem 1.2 is substantially easier and more direct than Wise’s argument.

**1.1. Proof outline and organization.** This paper is organized as follows. In Section 2, we review QF manifolds, convex thickenings, and geometric estimates during Dehn filling. We also extend the prior work of Baker and Cooper [4, 6] to prove the asymmetric combination theorem, Theorem 2.4, which roughly says that the convex hull of a union of convex pieces stays very close to one of the pieces. In Section 3, we prove several useful lemmas and characterize ubiquitous collections of surfaces using the notion of a compact pancake (see Definition 3.7). In Section 4, we assemble these ingredients to prove the following weaker version of Theorems 1.1 and 1.2.

**Theorem 1.4.** Let \( M = \mathbb{H}^3/\Gamma \) be a cusped hyperbolic 3–manifold. Let \( \alpha_1, \ldots, \alpha_n \) be a collection of slopes on \( \partial M \), with one slope per cusp. Then there is a ubiquitous set of cusped QF surfaces immersed in M, with the property that at least one boundary component of each surface is mapped to a multiple of each \( \alpha_i \).

It is worth observing that Theorem 1.4 already implies a weak version of Corollary 1.3, namely co-sparse cubulation. See Corollary 8.2 for a precise statement.

Here is the idea of the proof of Theorem 1.4. First, we perform a large Dehn filling on the cusps of \( M \) to produce a closed hyperbolic 3–manifold \( N \). Then, by results of Kahn and Markovic [24] and Agol [1], there is a finite cover \( \tilde{N} \) of \( N \) that contains a closed, embedded, almost geodesic QF surface. A small convex neighborhood of this surface is a compact QF manifold \( Q \subset \tilde{N} \), with strictly convex boundary. The preimages of the filled cusps form a collection \( \mathcal{W} \) of solid tori in \( \tilde{N} \). Gluing these onto \( Q \) and thickening gives a compact convex manifold \( Z \subset \tilde{N} \) with strictly convex boundary that is far from the core geodesics \( \Delta \subset \mathcal{W} \). Deleting \( \Delta \) gives a finite cover \( \tilde{M} \) of \( M \), with the property that the hyperbolic metric on \( M \setminus \mathcal{W} \) is very close to the hyperbolic metric on \( \tilde{N} \setminus \mathcal{W} \). It follows that \( \partial Z \) is also locally convex in \( \tilde{M} \), so \( Y = Z \setminus \Delta \) is a convex submanifold of \( \tilde{M} \). One now surgers
Lemma 2.1. Let $\partial M \subset Y$ along disks and annuli in $Y$ running out into the cusps of $\tilde{M}$ to produce an embedded, geometrically finite incompressible surface $F \subset Y$ without accidental parabolics. It follows that $F$ is quasi-Fuchsian, and the projection of $F$ into $M$ is an immersed QF surface with cusps. This use of the convex envelope $Z$ is similar to the method used by Cooper and Long [13] to show that most Dehn fillings of a hyperbolic manifold contain a surface subgroup.

To derive Theorem 1.1 from Theorem 1.4, we need to call upon several results and techniques developed by Baker and Cooper [6]. We review these results in Section 5. Given enough cusped surfaces, one can glue together finite covers of copies of convex thickenings of these surfaces, together with some finite covers of the cusps of $M$, to create a convex manifold $Z$ called a \textit{prefabricated manifold} (see Definition 5.5). This prefabricated manifold is immersed in $M$ by a local isometry, and each component of $\partial Z$ is closed and quasi-Fuchsian. Projecting $\partial Z$ down to $M$ yields a closed, immersed QF surface. A mild variation of this technique proves Theorem 1.2 in Section 7.

Finally, in Section 8, we explain how to combine Theorems 1.1 and 1.2 with results of Bergeron and Wise [7] and Hruska and Wise [22] to show that $\pi_1(M)$ acts freely and cocompactly on a CAT(0) cube complex.

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2. Background

This section lays out the definitions, conventions, and background material that will be used in subsequent arguments. Almost all of the results stated here are widely known and appear elsewhere in the literature. The one result with any novelty is Theorem 2.4, the \textit{asymmetric combination theorem}. This is a mild generalization of the convex combination theorem of Baker and Cooper [4, 6]. The generalized statement described here may be of some independent interest.

2.1. Convex and complete manifolds. A \textit{hyperbolic $n$–manifold} is a smooth $n$–manifold, possibly with boundary, equipped with a metric so that every point has a neighborhood that is isometric to a subset of hyperbolic space, $\mathbb{H}^n$. A connected hyperbolic $n$–manifold $M$ is \textit{convex} if every pair of points in the universal cover $\tilde{M}$ is connected by a geodesic. It is \textit{complete} if the universal cover is isometric to $\mathbb{H}^n$.

We emphasize that the hyperbolic manifolds considered in this paper are not necessarily complete. On the other hand, all manifolds in this paper are presumed connected and orientable, unless noted otherwise. As we describe at the start of Section 4.1, disconnected manifolds are typically denoted with calligraphic letters.

The following facts are straightforward; see [4, Propositions 2.1 and 2.3].

\textbf{Lemma 2.1.} Let $M$ be a convex hyperbolic $n$–manifold. Then the developing map embeds $\tilde{M}$ isometrically into $\mathbb{H}^n$, and the covering transformations of $\tilde{M}$ extend to give a group $\Gamma \subset \text{Isom}(\mathbb{H}^n)$. Consequently, $M$ is isometric to a submanifold of $N = \mathbb{H}^n/\Gamma$, where $N$ is unique up to isometry.

If $M$ is convex and $f : M \to N$ is a local isometry into a hyperbolic $n$–manifold $N$, then $f$ is $\pi_1$–injective.

The geodesic compactification of $\mathbb{H}^n$ is the closed ball $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial \mathbb{H}^n$. In the main case of interest, $n = 3$, we will write $\partial \mathbb{H}^3 = S^2_+$. The limit set of a subset $A \subset \mathbb{H}^n$ is $\Lambda(A) = \overline{A} \cap \partial \mathbb{H}^n$. If $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ is a discrete group, then $\Lambda(\Gamma)$ is the limit set of an orbit $\Gamma x$, for an arbitrary $x \in \mathbb{H}^n$.

The convex hull of a set $A \subset \mathbb{H}^n$, denoted $\text{CH}(A)$, is the intersection of $\mathbb{H}^n$ and all the convex subsets containing $A$. If $M$ is a convex hyperbolic manifold, then by Lemma 2.1, $M$ isometrically embeds into a complete manifold $N = \mathbb{H}^n/\Gamma$. We define the \textit{convex core} of $M$ to be $\text{Core}(M) = \text{CH}(\Lambda(\Gamma))/\Gamma$. Then $\text{Core}(M) = \text{Core}(N)$, and $\text{Core}(M) \subset M \subset N$. 

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A convex hyperbolic $n$–manifold is geometrically finite if some (any) $\epsilon$–neighborhood of $\text{Core}(M)$ has finite volume. We will focus our attention on two special kinds of geometrically finite hyperbolic 3–manifolds.

2.2. Quasi-Fuchsian basics. A Fuchsian group is a discrete, torsion-free, orientation-preserving subgroup $\Gamma \subset \text{Isom}(\mathbb{H}^2)$, such that the quotient $S = \mathbb{H}^2/\Gamma$ has finite area. We call $S$ a finite area hyperbolic surface. A Fuchsian group $\Gamma$ stabilizing a hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$ can also be considered a subgroup of $\text{Isom}(\mathbb{H}^3)$. Note that the convex core of a Fuchsian group is $\text{Core}(\mathbb{H}^3/\Gamma) = \mathbb{H}^2/\Gamma = S$, which has 3–dimensional volume 0.

A convex hyperbolic 3–manifold $M$ is called quasi-Fuchsian (or QF for short) if there is a finite-area hyperbolic surface $S$ such that $\text{Core}(M)$ has finite volume and is homeomorphic $S \times I$ or to $S$. To overcome this mild technical irritation, we define a convex 3–manifold by $Q(S) = \text{Core}(M)$ unless $S$ is Fuchsian, in which case $Q(S)$ is a small convex neighborhood of $S$. Thus $Q(S) \cong S \times I$ in all cases.

By Lemma 2.1, every quasi-Fuchsian 3–manifold $Q$ isometrically embeds into a complete 3–manifold $N \cong S \times \mathbb{R}$. We call $N$ the universal thickening of $Q$ (compare Section 2.3). We call $S \times \{0\} \subset N$ a quasi-Fuchsian surface. If $f: N \to M$ is a covering map of complete hyperbolic 3–manifolds, and $N$ is quasi-Fuchsian, the restriction $f: S \times \{0\} \to M$ gives an immersed quasi-Fuchsian surface in $M$. Note that immersed QF surfaces are automatically $\pi_1$–injective.

Let $S$ be a finite-area hyperbolic surface. A representation $\rho: \pi_1 S \to \text{Isom}(\mathbb{H}^3)$ is called type-preserving if $\rho(\gamma)$ is parabolic for any loop $\gamma$ encircling a puncture of $S$. If a loop $\gamma \in \pi_1 S$ is not homotopic into a puncture but $\rho(\gamma)$ is parabolic, we say that $\gamma$ is an accidental parabolic for $\rho$.

We will deduce that certain manifolds are QF via the following classical result.

**Theorem 2.2.** Let $N = \mathbb{H}^3/\Gamma$ be a complete hyperbolic 3–manifold. Suppose that there is a finite-area hyperbolic surface $S$ an isomorphism $\rho: \pi_1(S) \to \Gamma$, which is type-preserving with no accidental parabolics. Suppose also that there is a covering map $f: N \to M$, where $M$ is geometrically finite and $\text{Core}(M) \neq M$. Then $N$ is quasi-Fuchsian.

**Proof.** First, we check that $N$ is geometrically finite. Suppose not. Then Canary’s covering theorem [10] (first proved by Thurston and Bonahon [8] in the special case of surface groups) says that $M$ must be a complete finite volume 3–manifold, such that $\Gamma = \rho(\pi_1 S)$ is a virtual fiber subgroup of $\pi_1 M$. In this case $\Lambda(M) = \Lambda(\Gamma) = S^2_\infty$, hence $\text{Core}(M) = M$, which contradicts the hypotheses.

Since $N$ is geometrically finite and $\rho$ has no accidental parabolics, it is a standard fact that $N$ is quasi-Fuchsian. \qed

2.3. Convex thickenings. Let $M$ and $N$ be (possibly disconnected) hyperbolic 3–manifolds with $M \subset N$. We say $N$ is a thickening of $M$ if the inclusion $\iota: M \hookrightarrow N$ is a homotopy equivalence. If, in addition, each component of $N$ is convex, then $N$ is called a convex thickening of $M$.

If $M$ is a subset of a metric space $N$, the (closed) $r$–neighborhood of $M$ in $N$ is

$$\mathcal{N}_r(M; N) = \{x \in N : d(x, M) \leq r\}.$$  

We will omit the second argument of $\mathcal{N}_r(\cdot, \cdot)$ when it is clear from context, for instance when the ambient set is $\mathbb{H}^n$. If $M$ his a convex hyperbolic $n$–manifold, recall from Lemma 2.1 that the developing map gives an isometric embedding $\tilde{M} \to \mathbb{H}^n$, and identifies $\pi_1 M$ with a discrete subgroup $\Gamma \subset \text{Isom}(\mathbb{H}^n)$. We define the $r$–thickening of $M$ to be

$$Th_r(M) = \mathcal{N}_r(\tilde{M}; \mathbb{H}^n) / \Gamma,$$

and set

$$Th_\infty(M) = \bigcup_{r > 0} Th_r(M) = \mathbb{H}^n / \Gamma.$$  

By construction, the universal thickening $Th_\infty(M)$ is complete.

A surface $S$ in a hyperbolic 3–manifold $M$ is locally convex if for each $x \in S$ there is a flat disk $D \subset M$ such that $D \cap S = x$. If $\kappa > 0$, then $S$ is $\kappa$–convex if every piecewise smooth curve in $S$ has geodesic curvature at least $\kappa$ in $M$. Thus $\kappa$–convexity implies local convexity. The following is easy:
Lemma 2.3. There is a continuous, monotonically increasing function $\kappa : \mathbb{R}_+ \to (0, 1)$ such that if $X \subset \mathbb{H}^3$ is a closed convex set and $x > 0$, then $\partial \text{Th}_x(X)$ is $\kappa(x)$-convex. □

The next result is a variant of the convex combination theorem of Baker and Cooper [4, 6].

Theorem 2.4 (Asymmetric combination theorem). For every $\epsilon > 0$, there is $R = R(\epsilon) > 8$ such that the following holds. Suppose that

1. $Y = Y_0 \cup \ldots \cup Y_m$ and $M = M_0 \cup \ldots \cup M_m$ are connected hyperbolic 3–manifolds.
2. $M_i$ and $Y_i$ are convex hyperbolic 3–manifolds, such that $Y_i$ is a thickening of $M_i$.
3. $Y_i \supset \text{Th}_R(M_i)$.
4. If $Y_i \cap Y_j \neq \emptyset$, then $i = j$ or $i = 0$.
5. Every component of $Y_i \cap Y_j$ contains a point of $M_i \cap M_j$.

Then $\text{CH}(M) \subset N_r(M_0; Y) \cup N_R(M_1 \cup \ldots \cup M_m; Y)$.

Theorem 2.4 strengthens [6, Theorem 1.3] in two small but useful ways. First, it allows a larger number of convex pieces to combine. Second, it records the conclusion that, far away from the pieces $M_1, \ldots, M_m$, the convex hull of $M$ must be $\epsilon$–close to $M_0$.

Proof. First, we verify that

$$\text{CH}(M) \subset \text{Th}_8(M_0) \cup \ldots \cup \text{Th}_8(M_m) \subset Y.$$ 

When $m = 1$, this is a special case of [6, Theorem 1.3]. (In that theorem, the pair $M_0, M_1$ are denoted $M_1, M_2$.) Indeed, hypotheses (C1), (C2) of that theorem are restated in (1), (2). Hypotheses (C3), (C4), (C5) of that theorem are all implied by (2) and (3). Hypothesis (C6) of that theorem is restated in (5). Thus the desired conclusion holds when $m = 1$.

The proof of [6, Theorem 1.3] combines a lemma about thin triangles with [4, Theorem 2.9] to conclude that $r = 8$ is a sufficient thickening constant. Thus almost all of the work is performed in [4, Theorem 2.9].

For $m > 1$, we observe that the proof of [4, Theorem 2.9] goes through verbatim when $M_1$ is replaced by the disjoint union $M_1 \sqcup \ldots \sqcup M_m$. Note that condition (4) guarantees that $Y_1, \ldots, Y_m$ are pairwise disjoint, hence $M_1, \ldots, M_m$ are as well. This concludes the proof that $\text{CH}(M) \subset Y$.

To prove the stronger containment claimed in the theorem, we need the following easy lemma.

Lemma 2.5. For every $\epsilon > 0$, there is $\tilde{R} = \tilde{R}(\epsilon) \geq 1$ such that the following properties hold.

1. For every triangle in $\mathbb{H}^3$ with vertices $a, b, c$, the geodesic $[a, c]$ satisfies $[a, c] \subset N_{\tilde{R}}([a, b]) \cup N_{\tilde{R}}([b, c])$.
2. Every (skew) quadrilateral in $\mathbb{H}^3$ with vertices $a_1, b_1, b_2, a_2$ satisfies $[a_1, a_2] \subset N_{\tilde{R}}([a_1, b_1]) \cup N_{\tilde{R}}([a_2, b_2]) \cup N_{\tilde{R}}([b_1, b_2])$.

Proof. For (1), let $\delta = \ln(1 + \sqrt{2}) < 1$ be the hyperbolicity constant of $\mathbb{H}^3$. Thus, by the $\delta$–thin triangles property, we have $[a, c] \subset N_{\delta}(\mathbb{H}(a, b)) \subset N_{\delta}(\mathbb{H}(b, c))$. For larger values $\tilde{R} \geq 1$, the geodesic $\text{sub-segment} [a, c] \subset N_{\tilde{R}}([a, b])$ must be contained in a smaller and smaller neighborhood of $[b, c]$. Thus some $\tilde{R} = \tilde{R}(\epsilon)$ satisfies (1).

Conclusion (2) follows from (1) by triangle inequalities. □

We now return to the proof of Theorem 2.4. For any $\epsilon > 0$, let $R = R(\epsilon) = \hat{R}(\epsilon) + 16$, where $\hat{R}(\epsilon)$ is the function of Lemma 2.5. For every $x > 0$, let $Z_r = N_r(M_1 \cup \ldots \cup M_m; Y)$. Then we have already shown

$$\text{CH}(M) \subset \text{Th}_8(M_0) \cup \ldots \cup \text{Th}_8(M_m) = \text{Th}_8(M_0) \cup Z_8.$$

Since $R > 8$, this implies

$$\text{CH}(M) \cap Z_R \subset \text{CH}(M) \cap Z_8 \subset \text{CH}(M \cup (Z_8 \cup \text{Th}_8(M_0))).$$

where $Z_8 \cap \text{Th}_8(M_0) = \text{Th}_8(M_0 \cap M_1) \cup \ldots \cup \text{Th}_8(M_0 \cap M_m)$ is the thickened intersection.
Let $x \in CH(M) \setminus Z_R$. Our goal is to show that $x \in N_\ell(M_0)$. The above characterization of $CH(M) \setminus Z_R$, combined with a theorem of Carathéodory [31, Proposition 5.2.3], implies that $x$ lies in a geodesic 3–simplex $\Delta$ whose vertices $a_0, \ldots, a_3$ are in $M_0 \cup (Z_8 \cap \Tha(M_0))$. Let $b_i$ be the point of $M_0$ closest to $a_i$. Then $d(a_i, b_i) \leq 8$. In fact, if $a_i \neq b_i$, we must have if $a_i \notin M_0$, hence $a_i \in Z_8$, which means that $[a_i, b_i] \subset Z_{16}$.

Consider how far the segment $[b_i, b_j]$ can be from $[a_i, a_j]$. If $a_i \neq b_i$ but $a_j = b_j = c$, Lemma 2.5(1) gives
\[ [a_i, a_j] \subset N_{R}([a_i, b_i]) \cup N_{\epsilon/4}(b_i, b_j) \subset N_{R}(Z_{16}) \cup N_{\epsilon/4}([b_i, b_j]) = Z_{R} \cup N_{\epsilon/2}([b_i, b_j]). \]
Similarly, if $a_i \neq b_i$ and $a_j \neq b_j$, Lemma 2.5(2) gives
\[ [a_i, a_j] \subset N_{R}([a_i, b_i]) \cup N_{R}([a_j, b_j]) \cup N_{\epsilon/2}([b_i, b_j]) \subset N_{R}(Z_{16}) \cup N_{R}(Z_{16}) \cup N_{\epsilon/2}([b_i, b_j]) = Z_{R} \cup N_{\epsilon/2}([b_i, b_j]). \]
Let $\Delta'$ be the simplex in $M_0$ with vertices $b_0, \ldots, b_3$. Then the corresponding sides of $\Delta$ and $\Delta'$ either lie in $Z_R$ or are $\epsilon/2$ fellow-travelers. Since $x \in \Delta \setminus Z_R$, it follows that $d(x, \Delta') < \epsilon$. But the convex manifold $M_0$ contains $\Delta'$, hence $x \in N_\epsilon(M_0)$ as desired. $\Box$

2.4. Cusps, tubes, and Dehn filling. A cusped hyperbolic 3–manifold is one that is complete, non-compact, and with finite volume. Every cusped manifold $M$ can be decomposed into a compact submanifold $A$ and a disjoint union of horocusps. Here, a (3–dimensional, rank–2) horocusp is $C = B/\Gamma$, where $B \subset \h^3$ is a horoball and $\Gamma \cong \mathbb{Z} \times \mathbb{Z}$ is a discrete group of parabolic isometries that preserve $B$. The boundary $\partial C = \partial B/\Gamma$ is called a horotorus.

A tube is a compact, convex hyperbolic solid torus. Any tube $W$ contains exactly one closed geodesic, called the core curve and denoted $\delta(W)$. A round tube is a uniform $r$–neighborhood about its core curve.

Given a hyperbolic manifold $M$ and $\epsilon > 0$, the $\epsilon$–thick part $M^{2\epsilon}$ is the set of all points whose injectivity radius is at least $\epsilon/2$. The $\epsilon$–thin part is $M^{\leq \epsilon} = M \setminus M^{2\epsilon}$. A famous consequence of the Margulis lemma is that there is a uniform constant $\mu_3$ such that for every $\epsilon \leq \mu_3$ and every complete hyperbolic 3–manifold $M$, the thin part $M^{\leq \epsilon}$ is a disjoint union of horocusps and round tubes.

Let $M$ be a cusped hyperbolic 3–manifold. Given a horocusp $C \subset M$, and a slope $\alpha$ on $\partial C$, the length of $\alpha$, denoted $\ell(\alpha)$, is the length of a Euclidean geodesic representative. The normalized length of $\alpha$ is the quantity $\hat{L}(\alpha) = \ell(\alpha)/\sqrt{\text{area}(\partial C)}$, which is left unchanged when $C$ is expanded or contracted. The definitions of $\ell(\alpha)$ and $\hat{L}(\alpha)$ extend linearly to non-primitive homology classes in $H_1(\partial C)$.

For a slope $\alpha$ on $\partial C$, Dehn filling $M$ along $\alpha$ is the process of removing a horocusp $C$ and gluing in a solid torus $W$ so that the meridian disk is mapped to $\alpha$. The resulting 3–manifold is denoted $M(\alpha)$. For an integer $k > 1$, Dehn filling $M$ along $k\alpha$ produces a 3–orbifold with base space $M(\alpha)$ and singular locus of order $k$ along the core curve of the added solid torus $W$. The same definition applies to Dehn fillings of $M$ along multiple horocusps.

Thurston showed that the change in geometry under Dehn filling is controlled by the length of a filling slope [37]. Hodgson and Kerckhoff made this control much more quantitative [20, 21]. The following theorem, building on their work, is essentially due to Brock and Bromberg [9].

**Theorem 2.6.** Let $\epsilon > 0$, $\kappa > 0$, and $J > 1$ be constants, where $\epsilon \leq \mu_3$. Then there exists a number $K = K(\epsilon, \kappa, J)$ such that the following holds for every cusped hyperbolic 3–manifold $M$.

Let $C_1, \ldots, C_n$ be a disjoint collection of horocusps, where each $C_i$ is a component of $M^{\leq \epsilon}$. Let $A = M \setminus \bigcup C_i$. Let $\alpha_1, \ldots, \alpha_n$ be (possibly non-primitive) homology classes on $\partial C_1, \ldots, \partial C_n$. Then, for any Dehn filling in which each $\alpha_i$ satisfies $L_0(\alpha_i) \geq K \sqrt{n}$, we have the following.

1. $N = M(\alpha_1, \ldots, \alpha_n)$ has a complete hyperbolic metric, in which the cores $\delta_1, \ldots, \delta_n$ of the added solid tori are closed geodesics.
(2) There is a diffeomorphism \( \varphi: M \to N \setminus (\delta_1 \cup \ldots \cup \delta_n) \), whose restriction to \( A \) is \( J \)-bilipschitz.

(3) Let \( S \subset A \) be a surface. If \( S \) is \( \kappa \)-convex, then the image \( \varphi(S) \) is \( \kappa/2 \)-convex. Conversely, if \( \varphi(S) \) is \( \kappa \)-convex, then \( S \) is \( \kappa/2 \)-convex.

Proof. Brock and Bromberg proved the same result under hypotheses on \( N \) rather than \( M \) (one needs to assume that the total length of the cores \( \delta_1, \ldots, \delta_n \subset N \) is sufficiently small). See [9, Theorem 1.3] for the \( J \)-bilipschitz diffeomorphism and [9, Corollary 6.10] for the control of geodesic curvature. Using estimates by Hodgson and Kerckhoff [20], Magid translated their result into hypotheses on normalized length in \( M \) [26, Theorem 1.2]. As Hodgson and Kerckhoff clarify in [21], the methods apply equally well to a simultaneous orbifold filling of multiple cusps, so long as each \( \alpha_i \) has normalized length \( \hat{L}(\alpha_i) \geq K\sqrt{n} \). See the Remark containing [21, Equation (37)]. \( \square \)

We remark that the length cutoff \( K = K(\epsilon, \kappa, J) \) in Theorem 2.6 is independent of the manifold \( M \). The dependence of \( K \) on the constants \( \epsilon, \kappa, J \) is made explicit in forthcoming work of Futer, Purcell, and Schleimer [16]. We will not need this here. In fact, Theorem 2.6 is already stronger than what we need; see Remark 4.10.

2.5. Covers, lifts, and elevations. Throughout the paper, we will deal with immersed objects in hyperbolic 3–manifolds, as well as their preimages in (finite or infinite) covering spaces. This requires some careful terminology.

Suppose that \( X \) and \( M \) are manifolds, \( p: \hat{M} \to M \) is a covering map, and \( f: X \to M \) is an immersion. If \( f_*(\pi_1(X)) \subset p_*(\pi_1(M)) \), the map \( f \) lifts to \( \hat{f}: X \to \hat{M} \). Such a lift is determined by a local inverse to \( p \) at a point of \( f(X) \). We call \( \hat{f} \) the lift of \( f \) and the image \( \hat{f}(X) \) the lift of \( X \) in \( \hat{M} \).

More generally, if \( \pi: \tilde{X} \to X \) is the universal covering map, then \( f \circ \pi \) always has a lift \( \hat{f}: \tilde{X} \to \hat{M} \). Again, the lift is determined by a local inverse to \( p \) at a point of \( f(X) \). We call the image \( \hat{f}(\tilde{X}) \) an elevation of \( X \) in \( M \). If \( p: \hat{M} \to M \) is a finite cover, an elevation of \( X \) in \( \hat{M} \) is a lift of some finite cover of \( X \). However, if \( f \) is not 1–1, an elevation of \( X \) may fail to be a component of \( p^{-1}(X) \).

A subgroup \( H \subset G \) is called separable if \( H \) is the intersection of finite index subgroups of \( G \). The group \( G \) is called residually finite if \{1\} is separable, and subgroup separable or LERF if all finitely generated subgroups are separable. A deep observation of Scott [35] is that if \( X \) is compact and \( f: X \to M \) is an immersion such that \( f_*(\pi_1(X)) \) is separable in \( \pi_1(M) \), then there is a finite cover \( \hat{M} \) and a lift \( \hat{f}: X \to \hat{M} \) that is homotopic to an embedding. If \( X \) and \( M \) are convex hyperbolic manifolds, then no homotopy is needed.

Scott showed that the fundamental groups of surfaces are subgroup separable [35]. Agol [1] showed that the fundamental groups of closed hyperbolic 3–manifolds are subgroup separable, completing a program developed by Wise [38, 39]. Our argument in Section 4 uses Agol’s theorem, although this is mainly a matter of convenience; see Remark 4.11. The argument in Section 7 uses subgroup separability in surfaces, and draws on the previous work of Baker and Cooper that does the same.

3. Fat tubes, thin surfaces, and pancakes

This section lays out some elementary results that will be needed in the proof of Theorem 1.4. In Section 3.1, we explore the notion of quasi-Fuchsian manifolds with nice product structures, and show that geodesics in such manifolds are naturally classified into three types (Definition 3.5). In Section 3.2, we characterize ubiquitous collections of surfaces in a 3–manifold using the notion of a pancake (Definition 3.7). The advantage of this point of view is that pancakes are compact objects, hence are well-behaved under Dehn filling.

3.1. Nice product structures. A product structure on a quasi-Fuchsian manifold \( Q \cong S \times I \) is a diffeomorphism \( f: S \times [-1, 1] \to Q \). The two boundary components of \( Q \) are \( \partial_+ Q = f(S \times \{+1\}) \) and \( \partial_0 Q = f(S \times \{-1\}) \). The arc \( f(x \times [-1, 1]) \) is called vertical. A product structure determines a map \( \pi_1: Q \to S \) called horizontal projection. The midsurface of \( Q \) is \( f(S \times 0) \). The mid-surface is only well-defined up to isotopy, because it depends on the choice of product structure.
The constants in the following definition are convenient but somewhat arbitrary.

**Definition 3.1.** A properly embedded geodesic $\alpha$ in a convex hyperbolic manifold $M$ has a fat collar if any path $\gamma$ that starts and ends on $\alpha$ and has $t(\gamma) < 0.01$ is homotopic into $\alpha$. In particular, $\alpha$ must be length-minimizing on a scale up to 0.01.

A product structure on $Q \cong S \times I$ is nice if every vertical arc $\beta$ has length and curvature at most 0.01, and if the endpoints of $\beta$ meet $\partial Q$ almost orthogonally. Here, “almost orthogonal” means that tangent vectors to $\beta$ and $\partial Q$ have inner product less than 0.01.

The thickness of a convex QF manifold $Q$ with boundary $\partial_+ Q \sqcup \partial_- Q$ is

$$t(Q) = \max\{d(\partial_- Q, x), d(\partial_+ Q, x) : x \in \partial_\pm Q\}.$$

**Lemma 3.2** (Almost flat implies thin). Given $\epsilon > 0$ there is $\kappa > 0$, such that if $S$ is a QF surface with extrinsic curvature everywhere less than $\kappa$, then $t(Q(S)) < \epsilon$.

**Proof.** If $\kappa$ is small enough, every geodesic in $\tilde{S}$ is very close to a geodesic in $\mathbb{H}^3$ with the same endpoints. By Carathéodory [31, Proposition 5.2.3], the convex hull of $\tilde{S}$ is the union of tetrahedra with vertices in $\tilde{S}$. Hence every point in $\tilde{Q}(S)$ is very close to $\tilde{S}$. This implies $Q(S)$ is thin. $\square$

**Lemma 3.3.** There exists $\epsilon > 0$ such that every convex QF manifold $Q$ with $t(Q) < \epsilon$ has a nice product structure.

**Proof.** A product structure can be constructed on $Q$ by using a partition of unity to combine unit vector fields whose integral curves are geodesics and that are defined in balls of radius .001 that cover $Q$ and are almost orthogonal to $\partial Q$. The flow defined by this combined vector field can be reparameterized to give a nice product structure. Further details are left to the reader. $\square$

A 1–manifold $\alpha \subset Q$ is called unknotted if there is a product structure on $Q$ such that $\pi_h|_\alpha$ is injective. Because any two product structures on $Q$ are isotopic, a closed curve $\alpha \subset Q$ is unknotted if and only if it is isotopic to a simple closed curve in $\partial Q$. An arc $\alpha$ properly embedded in $Q$ is unknotted if and only if it properly isotopic to either a vertical arc, or to an arc in $\partial Q$. Geodesics in general QF manifolds can be knotted, but nice product structures preclude this:

**Lemma 3.4** (Unknottedness). Suppose $Q$ is a QF manifold with a nice product structure. Let $\alpha \subset Q$ be a geodesic with a fat collar. Then $\alpha$ is compact and unknotted in $Q$.

**Proof.** The fat collar about $\alpha$ prevents it from accumulating on itself inside $Q$ or traveling too deep into any cusps of $Q$. Thus $\alpha$ is compact.

Let $F$ be the union of all vertical arcs (with respect to the nice product structure) that contain a point of $\alpha$. If some vertical arc $\beta$ contains two points of $\alpha$, the niceness of the product structure and the fat collar about $\alpha$ imply that $\alpha$ is almost vertical. Thus $\alpha$ is injective, and $\pi_h|_\alpha$ is injective, hence $\alpha$ is unknotted by definition. $\square$

As a consequence, we have the following classification of geodesics in thin QF manifolds.

**Definition 3.5.** Let $Q \cong S \times I$ be a QF manifold with a nice product structure, and let $\text{Th}_\infty(Q)$ be the universal thickening of $Q$. Let $\delta \subset \text{Th}_\infty(Q)$ be an embedded geodesic with a fat collar, and let $\alpha = \delta \cap Q$. We say that $\delta$ is

- **skirting** if $\alpha = \emptyset$ or an interval whose endpoints are on the same component of $\partial Q$.
- **meridional** if $\alpha$ is an interval whose endpoints are on different components of $\partial Q$.
- **longitudinal** if $\alpha = \delta$ is a closed, unknotted geodesic in $Q$.

The terminology can be explained as follows. In the meridional case, $\partial_+ Q \setminus \delta$ has a puncture that is a meridian of $\delta$. In the longitudinal case, $\delta$ is isotopic into $\partial_+ Q$, hence removing it creates a pair of loops in $\partial_+ Q$ that are longitudes of $\delta$.
Lemma 3.6. Let \( R = \mathbb{H}^3 / \Gamma \cong S \times \mathbb{R} \) be a complete QF manifold, and let let \( \delta \subset R \) be an embedded geodesic with a fat collar. Suppose that some convex thickening \( Q \) of Core\((S)\) has a nice product structure. Then \( \alpha = \delta \cap Q \) is either empty or connected. Furthermore, \( \delta \) satisfies one of the three types enumerated in Definition 3.5. The type of \( \delta \) is independent of the choice of thickening of Core\((S)\).

Proof. Note that if \( \alpha \neq \emptyset \), it must be connected by convexity of \( Q \). In addition, \( \alpha \) is compact and unknotted by Lemma 3.4. Thus one of the above three possibilities must always hold.

Let \( \tilde{\delta} \) be an elevation of \( \delta \) to \( \tilde{R} = \mathbb{H}^3 \). This is a bi-infinite geodesic with endpoints \( x_{\pm} \). Let \( \Lambda = \Lambda(\Gamma) \subset \mathbb{S}^2_{\infty} \) be the limit set of \( \Gamma \). Then \( \delta \) is skirting if and only if \( x_{\pm} \) lie in the same component of \( \mathbb{S}^2_{\infty} \setminus \Lambda \); meridional if and only if \( x_{\pm} \) lie in different components of \( \mathbb{S}^2_{\infty} \setminus \Lambda \); and longitudinal if and only if \( x_{\pm} \in \Lambda \). This classification depends only on \( \Gamma \), hence is independent of the choice of thickening of Core\((S)\). \( \blacksquare \)

3.2. Pancakes ensure ubiquity.

Definition 3.7. For \( 0 < \eta < r \), a pancake is

\[
P(\eta, r) = N_{\eta}(D_r; \mathbb{H}^3),
\]

where \( D_r \) is a closed disk of radius \( r \) in a totally geodesic hyperbolic plane in \( \mathbb{H}^3 \). The points of \( \partial P(\eta, r) \) that are distance \( \eta \) from \( \partial D_r \) form an annulus called the vertical boundary, denoted \( \partial_V P(\eta, r) \). Meanwhile, \( \partial P(\eta, r) \setminus \partial_V P(\eta, r) \) consists of two disks, called the upper and lower boundary, and denoted \( \partial_{\pm} P(\eta, r) \).

Let \( X \) be a submanifold of \( \mathbb{H}^3 \). We say that \( X \) separates \( P \) if \( \partial_{-} P \) and \( \partial_{+} P \) are contained in different path-components of \( P \setminus X \), and \( X \) strongly separates \( P \) if \( \partial_{-} P \) and \( \partial_{+} P \) are contained in different path-components of \( \mathbb{H}^3 \setminus X \).

Lemma 3.8. Let \( \Pi_-, \Pi_+ \subset \mathbb{H}^3 \) be hyperbolic planes, such that \( d(\Pi_-, \Pi_+) = 2\eta > 0 \). Then there is a radius \( r = r(\eta) > 0 \) and a pancake \( P = P(\eta, r(\eta)) \) contained between \( \Pi_- \) and \( \Pi_+ \), such that every convex submanifold \( X \subset \mathbb{H}^3 \) that strongly separates \( P \) will also separate \( \Pi_- \) from \( \Pi_+ \).

![Figure 1. The construction of a pancake \( P = P(\eta, r) \) in Lemma 3.8.](image-url)
Proof. The pancake $P = P(\eta, r)$ is constructed as follows. Let $\gamma$ be the unique geodesic segment of length $2r$ connecting $\Pi_-$ to $\Pi_+$, and let $y$ be the midpoint of $\gamma$. Let $C \subset H^3$ be the double cone constructed by coning $y$ to $\partial \Pi_\pm$.

The disk $D_r$ will lie in the plane perpendicular to $\gamma$ at $y$. We choose a radius $r = r(\eta)$ large enough to ensure that the vertical boundary of $P(\eta, r) = N_r(D_r)$ lies entirely outside $C$. See Figure 1.

Let $X \subset H^3$ be a convex submanifold that strongly separates $P$. In particular, $X$ is disjoint from $\partial_\pm P$. Suppose, for a contradiction, that there is a point $x \in X \cap \Pi_\pm$. The geodesic line $\beta$ through $x, y$ must meet both $\partial_- P$ and $\partial_+ P$, hence contains points of $X$ between $\partial_- P$ and $\partial_+ P$. By convexity, the interval $\beta \cap X$ must intersect $\partial_\pm P$, since $x \in \beta \cap X$ is above $\partial_\pm P$. This contradicts the hypothesis that $X$ is disjoint from $\partial_\pm P$. Thus $X$ must be disjoint from $\Pi_\pm$.

To show that $X$ separates $\Pi_-$ from $\Pi_+$, consider a path $\beta \subset H^3$ from $\Pi_- \to \Pi_+$. Suppose, for a contradiction, that $\beta$ is disjoint from $X$. The geodesic segment $\alpha$ with the same endpoints as $\beta$ must intersect $P$ in an interval $I$, with $\partial I \subset \partial_\pm P$. Furthermore, $X \cap \alpha \subset P \cap \alpha = I$. By concatenating $\beta$ with the two components of $\alpha \setminus I$, we obtain a path from $\partial_- P$ to $\partial_+ P$ through $H^3 \setminus X$. This contradicts the hypothesis that $X$ strongly separates $P$. $\square$

The point of Lemma 3.8 is the following. If $M = H^3/\Gamma$ is a cusped hyperbolic manifold, and $\Pi \subset H^3$ is a generic hyperbolic plane, the image of $\Pi$ will be dense in $M$, and in particular will make arbitrarily deep excursions into the cusps $[32, 36]$. When we Dehn fill $M$, we can control geometry using Theorem 2.6, but this control only works in regions of $M$ that stay out of the very thin horocups. Thus, in proving ubiquity, we cannot directly control what happens to $\Pi_\pm$ in Dehn fillings. By contrast, the image of a pancake $P(\eta, r)$ in $M$ is compact, hence lies in $M^{\geq \epsilon}$ for some $\epsilon$. Thus we can use Theorem 2.6 to understand what happens to the pancake during Dehn filling. This is used in the proof of Theorem 4.1.

We will need to work with pancakes embedded in manifolds. If $f : H^3 \to M$ is a local isometry and $P \subset H^3$ is a pancake such that $f|_P$ is an embedding, we refer to the image $P_M = f(P) \subset M$ as a pancake in $M$. The upper and lower boundary $\partial_+ P_M$ and $\partial_- P_M$ are well-defined via $f$. A submanifold $X \subset M$ is said to (strongly) separate $P_M$ in $M$ if there are elevations $\tilde{X}$ of $X$ and $\tilde{P}_M$ of $P_M$ to the universal cover $\tilde{M}$, such that $\tilde{X}$ (strongly) separates $\partial_- \tilde{P}_M$ from $\partial_+ \tilde{P}_M$. Since universal coverings are regular, one may first choose a lift $\tilde{P}_M$ and then find an appropriate elevation $\tilde{X}$.

4. Drilling Kahn–Markovic surfaces

The main goal of this section is to prove Theorem 1.4. That result will follow immediately from the following reformulated statement.

Theorem 4.1. Let $M = H^3/\Gamma$ be a cusped hyperbolic 3–manifold. Let $\alpha_1, \ldots, \alpha_n$ be a collection of slopes on $\partial M$, with one slope per cusp. In addition, let $P = P(\eta, r)$ be a pancake in $H^3$. Then there are constants $k_i \in \mathbb{N}$ and an immersed QF surface $F \to M$, such that the following hold:

1. At least one boundary component of $F$ is mapped to $k_i \alpha_i$, for every $i$.

2. There is an elevation of $\text{Core}(F)$ to $H^3$ that strongly separates $P$.

Proof of Theorem 1.4, assuming Theorem 4.1. For any pair of disjoint planes $\Pi_\pm$, whose distance is $2\eta$, construct a pancake $P = P(\eta, r)$ as in Lemma 3.8. Let $F$ be a QF surface produced by Theorem 4.1. The elevation $\tilde{\text{Core}}(F) \subset H^3$ that strongly separates $P$ is a convex set, hence $\tilde{\text{Core}}(F)$ also separates $\Pi_-$ from $\Pi_+$. Thus the surfaces produced using Theorem 4.1 are ubiquitous. $\square$

It is worth recalling that Theorem 4.1 and its consequence in Theorem 1.4 already implies a weak form of cubulation for $\pi_1(M)$. See Corollary 8.2 below for details.

The proof of Theorem 4.1 consists of two halves: filling and drilling. First, we perform a long Dehn filling on $M$ along (multiples of) $\alpha_1, \ldots, \alpha_n$ to obtain a closed hyperbolic 3–orbifold $N$. For the sake of this outline, it helps to imagine that $N$ is a manifold. The Kahn–Markovic theorem gives a very thin immersed QF surface $S : N \to N$. We will study the intersection of a convex thickening...
of Core(S) with the union of Dehn filling tubes to build a “convex envelope” \( Z = Q \cup \mathcal{V} \), where \( \mathcal{V} \) consists of a subset of the tubes. This convex envelope is embedded in some cover of \( N \); for now it helps to imagine that no cover is needed.

In the second half of the proof, we surger the midsurface \( S \) of \( Q \), while taking care to stay within the convex envelope \( Z \). We then drill out the Dehn filling cores, recovering \( M = N \setminus \Delta \). This produces a (possibly disconnected) surface \( \mathcal{F} \subset Z \setminus \Delta \). The convexity of \( \partial Z \) ensures that the components of \( \mathcal{F} \) have convex cores contained in \( Z \setminus \Delta \), which implies they are quasi-Fuchsian.

We begin the proof in Section 4.1 by laying out the drilling portion of the argument. See Proposition 4.2 for a self-contained if somewhat lengthy statement. In Section 4.2, we lay out the Dehn filling argument, including repeated passage to covers, and incorporate Proposition 4.2 to complete the proof of Theorem 4.1. In Section 4.3, we sketch how the proof of Theorem 4.1 can be modified to avoid using several large hammers.

### 4.1. Drilling a quasi-Fuchsian surface.

We employ the following convention introduced by Baker and Cooper [6]. From now until the start of Section 8, calligraphic letters will always denote disjoint unions of objects (typically finitely many objects). The corresponding Roman letters denote the individual components. For instance, in the following proposition, \( \mathcal{V} \) denotes a disjoint union of tubes in a manifold \( N \), whereas \( \mathcal{V} \) is a single tube forming a component of \( \mathcal{V} \).

**Proposition 4.2.** Suppose that \( N \) is a complete hyperbolic 3–manifold and \( Q \subset N \) is a compact, embedded QF submanifold with a nice product structure. Suppose that \( \mathcal{V} \subset N \) is a disjoint union of tubes, such that every tube \( V \subset \mathcal{V} \) intersects \( Q \) in a single component of intersection and \( Q \cap \delta(V) \) is empty or has a fat collar. Let \( Z = Q \cup \mathcal{V} \). Suppose \( \Delta \subset N \) is a geodesic 1–manifold such that \( \Delta \cap Z \) is the disjoint union of the cores of \( \mathcal{V} \).

Suppose that \( M \) is a complete hyperbolic manifold with a diffeomorphism \( \phi : M \to N \setminus \Delta \), such that \( Y = \phi^{-1}(Z \setminus \Delta) \) is convex and has finite volume in the hyperbolic metric on \( M \).

Then \( M \) contains a (possibly disconnected) embedded surface \( \mathcal{F} \). Each component \( F \subset \mathcal{F} \) is quasi-Fuchsian, with Core(\( F \)) \( \subset Y \). The cusps of \( \mathcal{F} \) correspond to the components of \( \Delta \) meeting \( Q \), as follows. For each meridional geodesic \( \delta \subset \Delta \), one cusp of \( \mathcal{F} \) will be a meridian of \( \delta \). For each longitudinal geodesic \( \delta \subset \Delta \), two cusps of \( \mathcal{F} \) will be longitudinal of \( \delta \). Skirting geodesics do not contribute cusps of \( \mathcal{F} \).

Finally, suppose that \( P_M \subset M \) is an embedded pancake, such that \( \mathcal{V} \cap \phi(P_M) = \emptyset \) and \( Q \) strongly separates \( \phi(P_M) \) in \( N \). Then there is a component \( F \subset \mathcal{F} \) whose convex core strongly separates \( P_M \).

In the statement of the proposition, \( \phi(P_M) \subset N \) is diffeomorphic but not necessarily isometric to a pancake. Since the definition of (strongly) separating a pancake is purely topological, the statement that \( Q \) strongly separates \( \phi(P_M) \) is unambiguous.

The surface \( \mathcal{F} \) in the statement of Proposition 4.2 is constructed as follows. The tubes of \( \mathcal{V} \) can be subdivided into three types, according to how their cores intersect \( Q \). (See Definition 3.5.) For each type of tube, we will perform a local isotopy of the midsurface \( S \) of \( Q \). After this isotopy, we let \( \mathcal{F} = S \setminus \Delta \). Most of the proof is devoted to verifying that \( \mathcal{F} \) has all the desired properties; this verification is broken up into a number of claims.

**Proof of Proposition 4.2.** Let \( V \) be a tube component of \( \mathcal{V} \), let \( \delta = \delta(V) \) be the core of \( V \), and let \( \alpha = \delta \cap Q \). If \( H = Q \cap V \) is non-empty, then it is connected and convex, hence Lemma 2.1 implies \( \pi_1 H \) is isomorphic to a subgroup of \( \pi_1 V \cong \mathbb{Z} \). If \( \pi_1 H \cong \mathbb{Z} \), then \( H \) is a (convex) tube, hence \( \delta \subset H \subset Q \). Otherwise, \( \pi_1 H \cong \{1\} \) and \( H \) is a convex ball, hence \( \alpha = \delta \cap H \) is empty or an arc. In every case where \( \alpha \neq \emptyset \), it is compact and unknotted in \( Q \) by Lemma 3.4.

The universal thickening \( \text{Th}_{\infty}(Q) \) is a cover of \( N \). If \( \alpha = \delta \cap Q \) is non-empty, the isometric inclusion \( Q \hookrightarrow \text{Th}_{\infty}(Q) \) determines a unique complete geodesic \( \hat{\delta} \subset \text{Th}_{\infty}(Q) \) extending \( \alpha \). Following Definition 3.5, we call \( V \subset N \) a skirting tube, meridian tube, or longitude tube according to the type of \( \alpha = \hat{\delta} \cap Q \).
Let $S$ be the midsurface of $Q$ with respect to the chosen nice product structure. For each type of tube, we will perform an isotopy of $S$ in $H$. This can be done independently for each tube $V \subset \mathcal{V}$.

**Skirting tube:** If $\alpha \neq \emptyset$, then it has both endpoints on the same component of $\partial Q$. Isotop $S$ inside $H$ so it is disjoint from $\alpha$. If $\alpha = \emptyset$, no isotopy is needed. In either case, $Q \cup V$ is homeomorphic to $Q$ with $V$ glued onto the boundary along a disk.

There is a disk $\Sigma(V) \subset Q \cap N$, with $\partial \Sigma(V) \subset \partial Q \cap \partial V$, and with $\alpha \cap \Sigma(V) = \emptyset$. For future usage, we isotop each of $Q$ and $V$ to its own side of $\Sigma(V)$. (After this isotopy, $Q$ and $V$ may no longer be convex. However, the topological setup where $V$ is glued to $Q$ along $\Sigma(V)$ will be helpful below.)

**Meridian tube:** In this case, $H \cap \partial Q$ contains two disks $D_\pm(V) \subset \partial_\pm Q$, each containing an endpoint of $\alpha$, and each serving as a compressing disk for $V$. Thus, after an isotopy of $S$ inside $H$, we may assume $S \cap \delta$ consists of one transverse intersection point. In this case, $Q \cup V$ is homeomorphic to the manifold obtained by attaching a 1–handle to $Q$ along $D_\pm(V)$. Define $\Sigma(V) = (D_-(V) \cup D_+(V)) \setminus \delta$.

**Longitude tube:** In this case, $H$ is a tube that contains the unknotted circle $\alpha = \delta(V)$. As in the proof of Lemma 3.4, the vertical arcs in $Q$ meeting $\alpha$ form an embedded annulus $A(V) \subset H$, which is properly embedded in $Q$ and contains $\alpha$. This annulus has one boundary component in each component of $\partial_\pm Q$. After a vertical isotopy of $S$ in $H$, we may assume that $S$ contains $\alpha$.

In the longitudinal case, $Q \cup V$ is homeomorphic to $Q$. For future usage, we take two parallel copies of $A(V)$, denoted $A_0(V)$ and $A_1(V)$, and isotop the solid torus $V$ inward, until $H = Q \cap V$ is the portion of $Q$ contained between $A_0(V)$ and $A_1(V)$. After this isotopy, $V$ or $H$ may no longer be convex, but $H$ will still contain the closed geodesic $\delta$. Define $\Sigma(V) = A_0(V) \cup A_1(V)$.

Define $\Sigma(V) = \bigcup_{V \subset \mathcal{V}} \Sigma(V)$ to be the union of planar surfaces constructed above.

After performing the above isotopy of $S$ for each component of $\mathcal{V}$, the midsurface $S \subset Q$ has the following properties. If $\delta$ is the core of a longitude tube, then $\delta \subset S$. If $\delta$ is the core of a meridian tube, then $\delta$ intersects $S$ once transversely. If $\delta$ is the core of a skirting tube, then $\delta \cap S = \emptyset$.

Set $M = N \setminus \Delta$. Then $F = S \setminus \Delta$ is a surface properly embedded in $M$, which will be disconnected if and only if the union of the longitude tubes separates $S$.

Observe that $F$ has punctures of two kinds. Each meridian tube gives rise to one puncture of $F$, whose slope is a meridian of a component of $\Delta$. Each longitude tube gives rise to two punctures of $F$, whose slopes are longitudes of a component of $\Delta$. The skirting tubes do not contribute punctures.

From now on, we focus attention on a component $F \subset \mathcal{F}$. The following sequence of claims shows that $F$ has all the required properties.

**Claim 4.3.** $F$ is incompressible in $M$. Consequently, the inclusion–induced map $\rho: \pi_1 F \to \pi_1 M$ is a faithful, type-preserving representation.

Suppose otherwise. Then there is a compressing disk $D \subset M$, with $D \cap F = \partial D = \alpha$, an essential simple closed curve on $F$. Since $F \subset S$ and $S$ is incompressible in $N$, there must be a disk $D' \subset S$ with $\partial D' = \alpha$. This disk $D' \subset S$ cannot contain any component of $\Delta$, because each component is a geodesic in $N$. Thus $D'$ meets each component of $\Delta$ transversely in at most one point. Since $D \subset N \setminus \Delta$ is disjoint from $\Delta$ by hypothesis, the 2–sphere $E = D \cup D'$ must meet each component $\delta \subset \Delta$ transversely in at most one point. But the hyperbolic manifold $N$ is irreducible, so $E$ is separating, hence $E \cap \delta = \emptyset$. It follows that $D' = E \cap S$ is disjoint from $\Delta$, hence $D' \subset F$, and so $\alpha$ is not essential in $F$. This contradiction proves the claim.

Recall that $Z$ is the component of $Q \cup \mathcal{V}$ containing $Q$. Let $Y = \varphi^{-1}(Z \setminus \Delta) \subset M$. By hypothesis, $Y$ is a convex submanifold of $M$.

**Claim 4.4.** $F$ has no accidental parabolics in $M$.

Suppose, for a contradiction, that a non-peripheral loop in $F$ is a parabolic in $M$. By Jaco’s Annulus Theorem ([23, Theorem VIII.13]; see also [13, Lemma 2.1]), there is an embedded annulus $B \subset M$ with one boundary component an essential loop $\alpha \subset F$ and the other boundary component $\beta \subset T$, where $T \subset M$ is a horotorus. Since $\alpha \subset F \subset Y$, and $Y$ is convex, $T$ must bound a
Claim 4.5. is not actually accidental.

Claim 4.6. is not actually accidental.

Claim 4.5. is not actually accidental.

Claim 4.6. is not actually accidental.
To show that \( \text{Core}(F) \) strongly separates \( P_M \), it remains to check that \( \text{Core}(F) \cap \partial_+ P_M = \emptyset \). Recall that the endpoints of \( \varphi(\beta) \) lie outside \( Z = Q \cup V \). Thus the endpoints of \( \beta \) lie outside \( Y = \varphi^{-1}(Z \setminus \Delta) \). On the other hand, the convex set \( \text{Core}(F) \) is contained in \( Y \). This \( \beta \) starts and ends outside \( \text{Core}(F) \). In the cover \( \hat{M}_F \), the path \( \hat{\beta} \) starts and ends outside \( \text{Core}(\hat{F}) \). Since \( \hat{\beta} \) was an arbitrary path in \( M_F \) from \( \partial_- \hat{P}_M \) to \( \partial_+ \hat{P}_M \), the conclusion follows.

This completes the proof of Proposition 4.2. \( \square \)

4.2. Dehn filling and covers. Here are the main steps of the proof of Theorem 4.1.

1. Given a set of slopes \( \alpha_1, \ldots, \alpha_n \) on cusps of \( M \), perform a long Dehn filling, resulting in a hyperbolic orbifold \( N = M(k\alpha_1, \ldots, k\alpha_n) \). The integer \( k \) is chosen large enough so that Theorem 2.6 preserves much of the geometry of \( M \) in the filling. The cusps of \( M \) get replaced by a union of \( \mathcal{W} \) of tubes. See properties (F1)–(F5) for details.

2. Map the pancake \( P \subset \mathbb{H}^3 \) into \( M \), and then into \( N \) via the bilipschitz map \( \varphi \). The result is a pancake \( P_N \).

3. Apply the Kahn–Markovic theorem [24] to find an immersed QF surface \( S \to M \) that closely fellow-travels the disk \( D \). This surface will have transverse, meridional intersections with each core curve, and will strongly separate \( P_N \).

4. Pass to a cover \( \hat{N} \) of \( N \) where a certain thickening \( Q_0 \) of \( \text{Core}(S) \) is embedded, and \( Q_0 \) intersects each tube of \( \mathcal{W} \) at most once. A finite cover with these properties exists by Agol’s work [1]; see also Remark 4.11.

5. Working in the cover \( \hat{N} \), apply Theorem 2.4 to show that the union of \( Q_0 \) and the tubes that intersect it has a convex thickening \( Z \), such that \( \partial Z \) closely fellow-travels \( \text{Core}(S) \) on the regions of interest. In particular, \( Z \) still separates the pancake \( P_N \).

6. Apply Proposition 4.2 to the convex envelope \( Z \subset \hat{N} \), recovering a disconnected QF surface \( \mathcal{F} \). This surface is embedded in the cover \( \hat{M} \) of \( M \) corresponding to the cover \( \hat{N} \) of \( N \).

The above outline deliberately omits any mention of quantitative constants that control thickness, embedded collars, and geodesic curvature. We now proceed to the full proof, with constants.

Proof of Theorem 4.1. Let \( P = P(\eta, r) \) be a pancake in \( \mathbb{H}^3 \). Without loss of generality, assume that \( 5n/4 \) is smaller than the constant \( \epsilon \) from Lemma 3.3. (Otherwise, make the pancake \( P \) thinner and apply the same proof.) Let \( \kappa = \kappa(\eta/4) \) be as in Lemma 2.3. Then, for every convex set \( X \subset \mathbb{H}^3 \), the boundary \( \partial \text{Th}_{\eta/4}(X) \) is \( \kappa \)-convex. In particular, the pancake \( P = P(\eta, r) = \text{Th}_{\eta/4}(\text{Th}_{3n/4}(D_r)) \) has \( \kappa \)-convex boundary.

By residual finiteness, there is a finite regular cover of \( M \) such that the (locally isometric) immersion \( P \to M \) lifts to an embedding in the cover. In this regular cover, all preimages of \( \alpha_i \) have the same form \( k_i \alpha_i \). Furthermore, any QF surface in a cover of \( M \), satisfying the desired conditions (1) and (2), will project to a QF surface in \( M \) with the same properties. Thus no generality is lost when we replace \( M \) by this finite cover. We now do so, keeping the name \( M \), and consider the embedded pancake \( P_M \subset M \) that lifts to \( P \subset \mathbb{H}^3 \). Thus \( \partial_+ P_M \) and \( \partial_- P_M \) are \( \kappa \)-convex, embedded surfaces in \( M \).

Let \( \epsilon < \mu_1 \) be small enough to ensure that \( P_M \subset M^{\leq \epsilon} \) and that \( d(P_M, \partial M^{\leq \epsilon}) \geq 1 \). Let \( R = R(\eta/4) \) be as in Theorem 2.4. Then there is a constant \( \epsilon' = e^{-1(R+2)} \epsilon \) with the property that in every horocusp \( C \subset M \), the \( \epsilon' \)-thin part of \( C \) is at least distance \( R + 1 \) away from \( M^{\geq \epsilon} \).

Let \( C_1, \ldots, C_n \) denote the horocusp components of \( M^{\leq \epsilon} \), and let \( C_i' \subset C_i \) be the corresponding horocusp components of \( M^{\leq \epsilon'} \). By the above choice of \( \epsilon' \), we have \( d(\partial C_i', \partial C_i) \geq R + 1 \). The horotori \( \partial C_i \) and \( \partial C_i' \) are 1–convex.
Let $A = M \setminus \bigcup C_i$ and $A' = M \setminus \bigcup C'_i$. Then we may think of $\alpha_i$ as a slope on either $\partial C_i$ or $\partial C'_i$, with the same normalized length. For a large integer $k$, let $N = M(k\alpha_1, \ldots, k\alpha_n)$ be a closed hyperbolic orbifold obtained by Dehn filling on $M$. More precisely, $k \geq 1$ needs to be large enough so that Theorem 2.6 ensures a diffeomorphic embedding $\varphi: A' \to N$ with the following properties.

(F1) If $S \subset A'$ is a $\kappa$-convex surface, then $\varphi(S)$ is $\kappa/2$-convex. If $\varphi(S) \subset \varphi(A')$ is a $\kappa$-convex surface, then $S$ is $\kappa/2$-convex.

(F2) The image $\varphi(P_M)$ is a convex ball in $N$, with $\kappa/2$-convex boundary.

(F3) There is a thinner pancake $P_N = P(3n/4, r_N)$ isometrically embedded in $\varphi(A')$, so that $P_N$ separates $\varphi(\partial P_M)$ from $\varphi(\partial P_M)$.

(F4) For every $i \leq n$, the tori $\varphi(\partial C_i)$ and $\varphi(\partial C'_i)$ are $\kappa/2$-convex. As a consequence, there is a nested pair of (convex) tubes $W'_i \subset W_i$, with $\partial W'_i = \varphi(\partial C'_i)$ and $\partial W_i = \varphi(\partial C_i)$.

(F5) The lipschitz constants on $\varphi$ give $d(\partial W'_i, \partial W_i) \geq R + \eta$ and $d(\partial W_i, \partial P_N) \geq \eta$.

Each of the above statements is ensured by conclusions (2) and (3) of Theorem 2.6.

Let $\delta_i$ be the core of $W_i$. We define $W = W_1 \cup \cdots \cup \partial W_n$ and $W' = W'_1 \cup \cdots \cup W'_n$. Then each of $W$ and $W'$ is a disjoint union of convex tubes, with cores along $\Delta = \delta_1 \cup \cdots \cup \delta_n$.

Consider a covering map $p: \tilde{N} \to N$. Let $\Delta = p^{-1}(\Delta)$ and $\tilde{W} = p^{-1}(W)$. This defines a corresponding cover $\tilde{M} = \tilde{N} \setminus \Delta$, whose hyperbolic metric is lifted from $M$. The pancake $P_M \subset M$ has a lift $\tilde{P}_M \subset \tilde{M}$, namely the image of the pancake $P \subset \mathbb{H}^3$ stipulated in the hypotheses. Then the diffeomorphic image $\varphi(\tilde{P}_M) \subset \tilde{N}$ defines an isometric lift $\tilde{P}_N \subset \tilde{N}$, which still separates $\varphi(\partial \tilde{P}_M)$ from $\varphi(\partial \tilde{P}_M)$, as in condition (F3). We call $P_N \subset \tilde{N}$ the preferred lift of $P_N$ to $\tilde{N}$.

Applying the above construction to $\tilde{N} = \mathbb{H}^3$ gives a preferred lift $\tilde{P}_N \subset \mathbb{H}^3$.

**Claim 4.7.** There is a totally geodesic disk $D \subset \mathbb{H}^3$ with the following properties:

(D1) The disk $D$ intersects at least one elevation of each $\delta_i \subset N$.

(D2) Every intersection of $D$ with an elevation of $\delta_i$ is transverse.

(D3) There is a number $\zeta > 0$ such that $N_{\zeta + \eta/2}(D)$ separates the preferred lift $\tilde{P}_N \subset \tilde{N} = \mathbb{H}^3$.

This follows as a corollary of a result by Shah [36] and Ratner [32]: almost every geodesic plane $\Pi \subset \mathbb{H}^3$ has dense image in $N$. Hence, a small perturbation of the midplane of $\tilde{P}_N$ will contain a disk with the desired properties.

Define a pancake $\tilde{P} = N_\zeta(D) \subset \mathbb{H}^3$. Then, by (D3), the thicker pancake $\text{Th}_{\eta/2}(\tilde{P}) = N_{\zeta \eta/2}(D)$ separates $\tilde{P}_N \subset \mathbb{H}^3$. This thickened pancake $\text{Th}_{\eta/2}(\tilde{P})$ maps to $N$ by a local isometry.

So far, $\tilde{N}$ is a closed hyperbolic orbifold, with singular locus of order $k$ along the core of every tube $W_i \subset \tilde{W}$. By Selberg’s lemma, a finite regular cover $\tilde{N}$ of $\tilde{N}$ is a closed hyperbolic manifold. The preimage of $W$ in this cover, denoted $\tilde{W} \subset \tilde{N}$, is a disjoint union of tubes with non-singular core along the preimage $\tilde{\Delta}$ of $\Delta$. Then $\tilde{N}$ is a Dehn filling of $\tilde{M} = \tilde{N} \setminus \Delta$. Note that the meridian of every component of $\tilde{\Delta}$ is a primitive slope in $\tilde{M}$, which maps to $k\alpha_i \subset M$ for some $i$.

By residual finiteness, we may select $\tilde{N}$ so that the locally isometric immersion $\text{Th}_{\eta/2}(\tilde{P}) \to \tilde{N}$ lifts to an embedding in $\tilde{N}$, with image an embedded pancake $\text{Th}_{\eta/2}(\tilde{P}_N)$.

We will pass to finite covers of $N$ several more times, keeping the name $\tilde{N}$. Each newly constructed $\tilde{N}$ will be a cover of the previously constructed covers of $N$. Each time we pass to a cover, we keep the complete preimage of the nested sets $\Delta \subset W' \subset W$, denoted $\Delta \subset W' \subset W$, respectively. Each time, the manifold $\tilde{M} = \tilde{N} \setminus \tilde{\Delta}$ is a finite cover of $M$. In contrast with the complete preimage $W$, we keep only the preferred lift of the pancakes $P_M$ and $P_N$, denoted $\tilde{P}_M$ and $\tilde{P}_N$ respectively. Each time, the lifted diffeomorphism $\tilde{\varphi}: \tilde{M} \to \tilde{N} \setminus \tilde{\Delta}$ continues to satisfy (F1)–(F5). Combining (F3) with (D3), we have an embedded pancake $\tilde{P}_N \subset \tilde{N}$, with an embedded thickening $\text{Th}_{\eta/2}(\tilde{P}_N)$ that separates $\tilde{\varphi}(\partial \tilde{P}_M)$ from $\tilde{\varphi}(\partial \tilde{P}_M)$. See Figure 2.

By the Kahn–Markovic theorem [24], there is a ubiquitous collection of immersed QF surfaces in $\tilde{N}$, with arbitrarily small extrinsic curvature. By Lemma 3.2, a surface with small extrinsic curvature has a convex core with very small thickness. Thus their theorem implies
Claim 4.8. There is a quasi-Fuchsian covering \( \pi: S \times \mathbb{R} \rightarrow \hat{N} \), such that \( \text{Core}(S) \) has a convex thickening \( Q_0 = Q(S) \cong S \times I \) with the following properties.

(KM1) \( t(Q_0) < \eta/4 \).

(KM2) \( \pi: Q_0 \rightarrow \hat{N} \) has an elevation \( \tilde{Q}_0 \subset \hat{N} = \mathbb{H}^3 \), which strongly separates the preferred lift \( \tilde{P}_N' \).

(KM3) Every component of \( \pi^{-1}(\Delta) \) that intersects \( \pi^{-1}(\tilde{P}_N') \) is meridional in \( Q_0 \) (see Definition 3.5).

Property (KM1) holds by Lemma 3.2, because Kahn–Markovic surfaces are almost geodesic. Property (KM2) holds because these surfaces are ubiquitous. Property (KM3) holds because \( S \) can be chosen so that an elevation \( Q_0 = Q(S) \) lies arbitrarily close to the disk \( D \) of Claim 4.7, whose intersections with the elevations of \( \Delta \) are transverse.

By Agol’s theorem [1, Theorem 9.2], \( \pi_1(\hat{N}) \) is subgroup separable, hence we may pass to a finite cover of \( \tilde{N} \) in which \( \text{Th}_\eta(Q_0) \) is an embedded submanifold. We replace \( \hat{N} \) by this cover, retaining the name \( \hat{N} \). As described above, we keep the complete preimage of the unions of tubes \( W \) and \( W' \), and the single preferred lift of each pancake.

Applying subgroup separability again, we pass to a finite cover of \( \hat{N} \) such that \( \text{Th}_\eta(Q_0) \) lifts to the cover, and \( W \cap \text{Th}_\eta(Q_0) \) is empty or connected for every component \( \hat{W} \subset \hat{W} \). By convexity, this means \( \hat{W}' \cap \text{Th}_\eta(Q_0) \) is empty or connected for every component \( \hat{W}' \subset \hat{W}' \).

Let \( \hat{W}'_1, \ldots, \hat{W}'_m \) be the components of \( \hat{W}' \) with the property that \( \text{Th}_{\eta/2}(\hat{W}'_i) \cap Q_0 \neq \emptyset \). Let \( \hat{W}'_1, \ldots, \hat{W}_m \) be the corresponding components of \( \hat{W} \). We will apply Theorem 2.4 to \( Q_0 \) and these tubes. That is, \( Q_0 \) and \( \text{Th}_\eta(Q_0) \) will play the roles of \( M_0 \) and \( Y_0 \), respectively. For \( 1 \leq j \leq m \), the nested tubes \( \text{Th}_{\eta/2}(\hat{W}'_j) \) and \( \hat{W}_j \) will play the roles of \( M_j \) and \( Y_j \), respectively. Note that each of the above submanifolds of \( \hat{N} \) is convex. The tubes \( \hat{W}_j \) are disjointly embedded by the Dehn filling construction of \( N \). Furthermore, condition (F5) gives \( \hat{W}_j \supset \text{Th}_{R+\eta}(\hat{W}'_j) \). Thus all the hypotheses of Theorem 2.4 hold for \( R = R(\eta/4) \), and we have

\[
Z := \text{Th}_{\eta/4} \left( Q_0 \cup \text{Th}_{\eta/2}(\hat{W}'_1) \cup \ldots \cup \text{Th}_{\eta/2}(\hat{W}'_m) \right)
\subset \text{Th}_{\eta/4} \left( Q_0 \cup \text{Th}_{R+\eta/2}(\hat{W}'_1) \cup \ldots \cup \text{Th}_{R+\eta/2}(\hat{W}'_m) \right)
= \text{Th}_{\eta/2}(Q_0) \cup \text{Th}_{R+\eta/4}(\hat{W}'_1) \cup \ldots \cup \text{Th}_{R+\eta/4}(\hat{W}'_m)
\subset \text{Th}_{\eta/2}(Q_0) \cup \hat{W}_1 \cup \ldots \cup \hat{W}_m.
\]

Here, the first containment is by Theorem 2.4 and the second containment is by condition (F5).

Define \( Q = Z \cap \text{Th}_{\eta/2}(Q_0) \) and \( V = Z \cap (\hat{W}_1 \cup \ldots \cup \hat{W}_m) \). Then \( V \) is a disjoint union of tubes, each of which has connected intersection with \( Q \). Since \( t(Q_0) < \eta/4 \), and the thickening adds \( \eta/2 \) to each side, we have \( t(Q) < 5\eta/4 \). Thus, by the choice of \( \eta \) at the beginning of the proof, Lemma 3.3 implies \( Q \) has a nice product structure.

Recall that \( Z \) is the \((\eta/4)\) thickening of a convex set in \( \hat{N} \). Thus, by the definition of \( \kappa \) at the beginning of the proof, \( \partial Z = \partial(Q \cup V) \) is \( \kappa \)-convex. By construction, every component of \( \hat{W}' \cap V \) is properly contained inside \( Z \). By the definition of \( \hat{W}'_1, \ldots, \hat{W}'_m \subset V \), every tube of \( \hat{W}' \setminus V \) is disjoint
from $\text{Th}_{n/2}(Q_0)$, which means that these tubes lie entirely outside $Z$. Thus $\partial Z \subset \tilde{N} \setminus \tilde{W}' = \tilde{\phi}(\tilde{A}')$, where $\tilde{A}' \subset \tilde{M}$ is the region outside the horocusps on which $\tilde{\phi}$ has the desired metric properties. Therefore, by property (F1), it follows that $\tilde{\phi}^{-1}(\partial Z) \subset \tilde{M}$ is $\kappa/2$–convex.

Let $Y = \tilde{\phi}^{-1}(Z \setminus \Delta) \subset \tilde{M}$. Then the (finitely many) tubes of $\tilde{W}'$ that lie inside $Z$ are replaced by (finite volume) horocusps in $Y$. Thus $Y$ has finite volume. Furthermore, $\partial Y = \tilde{\phi}^{-1}(\partial Z)$ is $\kappa/2$–convex in the hyperbolic metric on $\tilde{M}$. Then an elevation of $Y$ to $\tilde{M}$ is a closed, connected subset of $\mathbb{H}^3$ with locally convex boundary, which means it is convex. Thus $Y$ is convex as well.

We have now checked that the submanifolds $Q, V, \tilde{\Delta} \subset \tilde{N}$ and $Y \subset \tilde{M}$ satisfy all the hypotheses required by Proposition 4.2. That proposition performs an isotopy of $S$ inside $Q$, after which every component of $\mathcal{F} = S \setminus \tilde{\Delta}$ is quasi-Fuchsian in the metric on $\tilde{M}$. To complete the proof of the theorem, we claim that one component $F \subset \mathcal{F}$ has all the desired properties.

**Claim 4.9.** There is a unique component $F \subset \mathcal{F}$ such that $F \cap \tilde{P}_M \neq \emptyset$. Furthermore,

1. For every slope $\alpha_i \subset M$, at least one cusp of $M$ maps to some multiple $k_i \alpha_i$.
2. There is an elevation of $\text{Core}(F)$ to $\mathbb{H}^3$ that strongly separates $P = \tilde{P}_M$.

Recall that by property (KM2), $Q_0$ strongly separates $\tilde{P}_N$, hence $Q \subset \text{Th}_{n/2}(Q_0)$ strongly separates $\mathcal{N}_{n/2}(\tilde{P}_N)$. Combining (F3) with (D3), as above, we conclude that $Q$ strongly separates $\tilde{\phi}(\tilde{P}_M)$. In particular, $Q \cap \tilde{\phi}(\tilde{P}_M) \neq \emptyset$ and $Q_0 \cap \tilde{\phi}(\tilde{P}_M) \neq \emptyset$.

By property (KM3), every component of $\tilde{\Delta}$ that intersects $\tilde{P}_N$ is meridional in $Q_0$. By Lemma 3.6, every component of $\tilde{\Delta}$ that intersects $\tilde{P}_N$ is also meridional in $\tilde{Q}$. In particular, $\tilde{\Delta} \cap \tilde{P}_N$ does not disconnect the midsurface $S$ of $Q$. Thus there is exactly one component $F \subset \mathcal{F}$ such that $F \cap \tilde{P}_M \neq \emptyset$.

By Proposition 4.2, every component of $\tilde{\Delta}$ that is meridional in $Q$ gives rise to a cusp of $\mathcal{F}$ that is a meridian of $\Delta$. By property (D2), the components of $\tilde{\Delta} \cap \tilde{P}_N$ include geodesics that project to every core $\delta_i \subset N$. Thus, for every slope $\alpha_i$ on $M$, the component $F \subset \mathcal{F}$ contains at least one cusp projecting to $k_i \alpha_i$. (The multiple $k_i$ incorporates the regular cover of $M$ constructed at the very beginning of the proof, as well as the manifold cover of the orbifold $N$ constructed using Selberg’s lemma.)

Finally, since $Q$ strongly separates $\tilde{\phi}(\tilde{P}_M)$, and $\tilde{W} \cap \tilde{\phi}(\tilde{P}_M) = \emptyset$ by property (F5), Proposition 4.2 implies that $\text{Core}(F)$ strongly separates $\tilde{P}_M$. Thus some elevation of $\text{Core}(F)$ to $\mathbb{H}^3$ strongly separates $P = \tilde{P}_M$. 

**4.3. Making do with less.** In this section, we outline how the proof of Theorem 4.1 can be modified to avoid appealing to Shah’s theorem [36], Agol’s theorem [1], or the work of Brock and Bromberg [9]. The upshot of Remarks 4.10–4.12 is that almost all of the technical work in the proof of Theorem 1.4 can be handled by soft classical arguments. The one “major hammer” needed for the proof is the Kahn–Markovic theorem [24].

**Remark 4.10.** In the proof of Theorem 4.1, our use of Brock and Bromberg’s Theorem 2.6 can be replaced by a softer appeal to Thurston’s results on geometric convergence under Dehn filling. Suppose $M$ is a finite volume hyperbolic manifold and $X \subset M$ is a compact submanifold obtained by removing a set of horocusps. Then the map $\text{Dev}(X) \rightarrow \text{Hom}(\pi_1 X, \text{PSL}(2, \mathbb{C}))$ that sends the developing map for an incomplete hyperbolic metric on $X$ to its holonomy is open. See Thurston [37, Theorem 5.8.2], Goldman [17], and Choi [11]; compare Cooper, Long, and Tillmann [12, Proposition 1.2]. It follows that for any sufficiently large Dehn filling $N$ of all the cusps of $M$, there is a diffeomorphic embedding of $X$ into $N$ that is close in the smooth topology to an isometry. In particular, such a map is $(1 + \epsilon)$–bilipschitz and the second derivative is close to 0. The control of second derivatives ensures that $\kappa$–convex surfaces stay convex.

**Remark 4.11.** In the proof of Theorem 4.1, our appeal to Agol’s theorem on subgroup separability [1] is convenient but not strictly necessary.
Picking up the proof after Claim 4.8, suppose we have found an immersed QF surface $S \to \hat{N}$, with immersed convex thickening $Q_0$. Then we may pass to an infinite cover $\hat{N} \to \hat{N}$, where $\text{Th}_8(Q_0)$ embeds and $\hat{W} \subset \hat{N}$ pulls back to a non-compact submanifold $\overline{W}$ with the following properties. For every component $W \subset \overline{W}$ such that $\text{Th}_\eta(Q_0) \cap W \neq \emptyset$, the intersection must be connected and $\overline{W}$ must be a compact tube. For concreteness, one could take $\overline{N}$ to be the cover corresponding to $\pi_1(Q_0)$ amalgamated with one extra loop (about some power of a core) for every skirting or meridional intersection $\text{Th}_\eta(Q_0) \cap \hat{W}$. The compactness of $Q_0$ ensures that only finitely many tubes intersect $\text{Th}_\eta(Q_0)$, hence $\pi_1(\overline{N})$ is finitely generated. In fact, the virtual amalgamation theorem of Baker and Cooper [4, Theorem 5.3] ensures that $\overline{N}$ is geometrically finite.

With this setup, Theorem 2.4 applies exactly as above to give a convex manifold $Z = Q \cup V$. Let $\Delta$ be the union of all the cores of $\overline{W}$, which may have non-compact components and infinitely many components. Then Proposition 4.2 applies to give an embedded QF surface in $\overline{M} = \overline{M} \setminus \Delta$. Since $Z \cap \Delta$ consists of finitely many closed geodesics, it still the case that $Y = \varphi^{-1}(Z \setminus \Delta)$ has finite volume. Note that $\overline{M}$ inherits its hyperbolic metric from $M$, and that Proposition 4.2 does not require any manifold except $Y$ to have finite volume. The surface $F \subset \overline{M}$ projects to an immersed QF surface in $M$, and Claim 4.9 verifies that $F$ has all the desired properties.

**Remark 4.12.** In the proof of Theorem 4.1, we use Claim 4.7 (which follows from the work of Shah [36] and Ratner [32]) to ensure that a certain region of a Kahn–Markovic surface hits all the Dehn filling cores transversely. This is used in Proposition 4.2 to produce a single component $F \subset \mathcal{F}$ with cusps along some multiple of every slope $\alpha_i$. However, even without Claim 4.7, we could focus on one slope $\alpha_i$ and apply Proposition 4.2 to find a connected QF surface $F_i$ with at least one cusp mapping to a multiple of $\alpha_i$. Moreover, the set of surfaces with this property is ubiquitous.

Given surfaces $F_1, \ldots, F_n$ realizing slopes $\alpha_1, \ldots, \alpha_n$, one could use the techniques of Baker and Cooper [4] to build a single surface $F$ immersed in $M$, such that every cusp of $F_i$ is covered by a cusp in $F$. As we explain in Section 5 below, such a surface $F$ can be obtained by gluing together subsurfaces of finite covers of the $F_i$. By the argument in Sections 6 and 7, the set of such surfaces is ubiquitous. This line or argument recovers Theorem 4.1 and proves Theorems 1.2 and 1.4 without relying on Claim 4.7.

### 5. Gluing and prefabrication

This section summarizes without proof some of the results of Baker and Cooper [6]. In what follows, $M$ is a complete finite volume hyperbolic 3–manifold and $Q$ is a finite volume QF manifold.

When working with 3–manifolds that contain surfaces with cusps, it is convenient to isotop everything so the cusps have compatible product structures. Suppose $B \subset \mathbb{H}^n$ is a horoball centered on a point $x \in \partial \mathbb{H}^n$ bounded by the horosphere $H = \partial B$. A *vertical ray* is a ray in $B$ that starts on $H$ and limits on $x$. Given $P \subset H$, the *set lying above* $P$ is called a *vertical set* and is the union, $V(P)$, of the vertical rays starting on $P$. If $P \subset H$ is convex, $V(P)$ is called a *thorn* and $P$ is called the *base of the thorn*. If $P = I \times \mathbb{R}$ is an infinite strip, $V(P)$ is a *slab*.

A hyperbolic $n$–manifold $E$ is an *excellent end* if it has finite volume and is isometric to $V/\Gamma$ for some vertical set $V \subset B$ and discrete group $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ preserving $V$. The *horospherical boundary* of $E$ is $\partial_H E = (V \cap H)/\Gamma$. An *excellent rank–1 cusp* is a 3–manifold $V/\Gamma$ where $V$ is a slab and $\Gamma$ is a cyclic group of parabolics preserving $V$.

A (possibly disconnected) hyperbolic manifold $M$ is *excellent* if $M = M^c \cup V_M$ where $M^c$ is compact and $M^c \cap V_M = \partial_H V_M$ and each component of $V_M$ is an excellent end. The pair $(M^c, V_M)$ is called an *excellent decomposition* of $M$. For example, an ideal convex polytope is excellent and the ends are thorns. In addition, a complete hyperbolic manifold with finite volume is excellent, since any ends of $M$ are horocusps. It is routine to show that if $S$ is a QF surface then $Q(S)$ has ends that are excellent rank–1 cusps, hence $Q(S)$ is excellent.

A compact, orientable surface properly embedded in a compact orientable 3–manifold is *essential* if it is incompressible and $\partial$–incompressible.
Definition 5.1. A surface $S$ embedded in an excellent 3–manifold $M = M^c \cup V$ is \textit{excellently essential} if each component of $S \cap V$ is an excellent annulus in the induced metric on $S$, and $S^c = S \cap M^c$ is a compact essential surface in $M^c$ with $\partial S^c \subset M^c$.

For example, if $Q(S)$ is embedded in $M$, then both components of $\partial Q(S)$ are excellently essential. Recall that a \textit{slope} on a torus is an isotopy class of essential simple closed curves. In view of the preceding, it makes sense to talk about the \textit{slope} of a excellently essential surface $S$ in a cusp of $M$, and the slope of a rank–1 cusp embedded in a rank–2 cusp.

Definition 5.2. An \textit{ideal spider} is an excellent convex hyperbolic 3–manifold $X$ with simply connected ends. Thus there is an excellent decomposition $X = B \cup \mathcal{L}$ such that $B$ is compact and convex, and each component of $\mathcal{L}$ is a thorn. The components of $\mathcal{L}$ are called \textit{legs} and $B$ is called the \textit{body}.

The definition implies that the holonomy of an ideal spider has no parabolics. A special case of an ideal spider with $k$ legs is a convex polytope with $k$ ideal vertices. In general, the body of a spider need not be simply connected. The following is immediate:

Proposition 5.3 (Proposition 3.3 in [6]). Suppose $M$ is a complete hyperbolic 3–manifold with finite volume and $Q_1, Q_2 \subset M$ are excellent QF submanifolds. Then $Q_1 \cap Q_2$ is excellent. If $Q_1$ and $Q_2$ have different slopes in every cusp of $M$, then each component of $Q_1 \cap Q_2$ is an ideal spider.

If $M$ and $N$ are excellent hyperbolic manifolds, a map $f: M \rightarrow N$ is \textit{excellent} if it is a local isometry and there are excellent decompositions with $f^{-1}(N^c) = M^c$. After a homotopy leaving $f|_{N^c}$ invariant, one can ensure that each vertical ray in $V_M$ maps to a vertical ray in $V_N$.

An \textit{immersed QF manifold} is a triple $(M, Q, f)$, where $f: Q \rightarrow M$ is an excellent map between excellent hyperbolic 3–manifolds and $Q$ is quasi-Fuchsian. Two immersed QF manifolds $(M, Q_1, f_1)$ and $(M, Q_2, f_2)$ have \textit{different slopes} if for every cusp $V_i \subset Q_i$, if $f_1(V_1)$ and $f_2(V_2)$ are in the same cusp of $M$ then they have different slopes. An \textit{immersed ideal spider} is $(M, R, p)$ where $M$ is an excellent 3–manifold, $R$ is an ideal spider, and $p: R \rightarrow M$ is an excellent map.

If $N$ is a submanifold of a cover of a hyperbolic manifold $M$, the restriction of the covering projection gives a local isometry $p: N \rightarrow M$, called the \textit{natural projection}. If $S$ is an immersed QF surface in $M$, the natural projection $Q(S) \rightarrow M$ is excellent. The following result generalizes Proposition 5.3 to immersed QF manifolds.

Theorem 5.4 (Theorem 3.4 in [6]). Suppose that $(M, Q_1, f_1)$ and $(M, Q_2, f_2)$ are immersed QF manifolds with different slopes. Suppose there are basepoints $q_i \in Q_i$, such that the basepoint $m = f_1(q_1) = f_2(q_2)$ is located in an excellent end of $M$.

Then there is a connected hyperbolic 3–manifold $P = \hat{Q}_1 \cup \hat{Q}_2$, where $p_i: \hat{Q}_i \rightarrow Q_i$ is a finite covering and $R = \hat{Q}_1 \cap \hat{Q}_2$ is an ideal spider with at least 2 legs. Thus $(Q_1, R, p_i|_R)$ is an immersed ideal spider.

The holonomy provides an identification of $\pi_1(M, m)$ with a Kleinian group $\Gamma \subset \text{Isom}(\mathbb{H}^3)$. Then the QF manifolds $Q_i$ have holonomy $\Gamma_i = (f_i)_*(\pi_1(Q_i, q_i)) \subset \Gamma$ and the holonomy of $R$ is $\Gamma_1 \cap \Gamma_2$.

Part of the point of Proposition 5.3 and Theorem 5.4 is that the spider $R = \hat{Q}_1 \cap \hat{Q}_2$ retains all the information for gluing $\hat{Q}_1$ to $\hat{Q}_2$. More evocatively, spiders know how to sew a web. This will be useful in our constructions, where we will want to glue further covers of $\hat{Q}_1$ and $\hat{Q}_2$. If the spiders lift to the covers, they retain all the necessary gluing instructions.

The goal in our proof of Theorem 1.1 is to take several immersed QF submanifolds of $M$ and some rank–2 cusps of $M$, and glue copies of finite covers of these manifolds to create a geometrically finite hyperbolic manifold $Z$ that has a convex thickening, such that $\partial Z$ consists of closed QF surfaces.

Definition 5.5 (Definition 1.1 in [6]). A \textit{prefabricated manifold} is a connected, metrically complete, finite-volume, hyperbolic 3–manifold

$$Z = \mathcal{C} \cup Q_1 \cup Q_2.$$
Each component of \( Q_i \) and of \( C \) is a convex hyperbolic 3–manifold called a **piece**. Each component of \( Q_i \) is a quasi-Fuchsian 3–manifold with at least one cusp. Each component of \( C \) is a horocusp. These pieces satisfy the following conditions for \( i \in \{1, 2\} \), and for each component \( C \) of \( C \):

(P1) \( Q_1 \cap C \) is the disjoint union of all the cusps in \( Q_1 \),

(P2) \( Q_1 \cap \partial C \) is an annulus with core curve \( \alpha_1(C) \),

(P3) \( \alpha_1(C) \) intersects \( \alpha_2(C) \) once transversely,

(P4) Each component of \( Q_1 \cap Q_2 \) intersects \( C \).

Note that by Proposition 5.3, each component of \( Q_1 \cap Q_2 \) is an ideal spider.

If pieces of a prefabricated manifold \( Z \) are sufficiently far apart, it will have a convex thickening.

**Corollary 5.6** (Corollary 1.4 in [6]). **Suppose** \( r \geq 8k \) **where** \( k = (|C| + |Q_1| + |Q_2| - 1) \), and suppose

(Z1) \( Z' = C' \cup Q'_1 \cup Q'_2 \) **is** a prefabricated manifold,

(Z2) \( Z = C \cup Q_1 \cup Q_2 \) **is** a prefabricated manifold **contained in** \( Z' \),

(Z3) \( Q'_i \) **is** a thickening of \( Q_i \),

(Z4) \( C' = Th_r(C) \),

(Z5) \( Q'_i \) contains an \( r \)-neighborhood of \( Q_i \setminus C \),

(Z6) **Every component** of \( Q'_1 \cap Q'_2 \) **contains** a point of \( Q_1 \cap Q_2 \).

Then \( Z \) **has** a convex thickening **that is a submanifold of** \( Z' \). Moreover, **every component** of \( \partial Z \) **is** compact and quasi-Fuchsian.

To achieve the metric separation needed for Corollary 5.6, one starts with a prefabricated manifold \( Z \) and replaces it by another prefabricated manifold \( Z' \), constructed from large finite covers of the pieces of \( Z \) with the property that all spiders used for the gluing lift. (Compare Section 2.5.) Choices of lifts of the spiders then determine \( Z' \). If the cover of each piece is regular (e.g. cyclic) and large enough, then one can choose lifts that are far apart.

Since a QF manifold is the product of a surface and an interval, constructing such covers reduces to questions about coverings of surfaces that contain various immersed surfaces (corresponding to the spiders). [6, Theorem 2.8] shows the corresponding covering spaces for these surfaces exist. The argument makes heavy use of subgroup separability arguments for subgroups of **surface groups** [35].

A covering space \( p: \tilde{F} \to F \) is called **conservative** if the surfaces \( F \) and \( \tilde{F} \) have the same number of boundary components. We frequently wish to take large conservative covers of surfaces with the property that certain immersed spiders lift to embedded spiders that are far apart. This is done using

**Theorem 5.7** (Theorem 0.1 of [5]; compare Theorem 9.1 of [28]). **Let** \( F \) **be** a compact, connected surface **with** \( \partial F \neq \emptyset \) **and** \( H \subset \pi_1 F \) **a finitely generated subgroup**. **Assume** that no loop representing an element of \( H \) **is** freely homotopic into \( \partial F \). **Given** a finite subset \( B \subset \pi_1 F \setminus H \), **there exists** a finite-sheeted conservative cover \( p: \tilde{F} \to F \) **and** a compact, connected, \( \pi_1 \)-injective subsurface \( S \subset \tilde{F} \) **such that** \( p_*(\pi_1 S) = H \) **and** \( p_*(\pi_1 \tilde{F}) \cap B = \emptyset \) **and** \( F \setminus S \) **is** connected.

The crucial ingredient for constructing a prefabricated manifold is a supply of QF surface subgroups with the property that for every cusp \( V \) of \( M \), there are (at least) two cusps of this collection that are contained in \( V \) and have different slopes. This is used to ensure property (P3). In turn, (P3) implies that \( \partial Z \) contains no accidental parabolics; see Corollary 5.6 and Proposition 7.2.

**6. Ubiquitous closed surfaces**

In this section, we prove Theorem 1.1. The proof is a modification of the proof of [6, Theorem 4.2]. In that proof, Baker and Cooper construct a QF surface \( \Sigma \) from a pair of (possibly disconnected) cusped surfaces \( J_1, J_2 \), with all components quasi-Fuchsian. The surface \( \Sigma \) is the boundary of a prefabricated manifold \( Z \), and is obtained by gluing together subsurfaces of the components of \( J_1 \) and \( J_2 \), together with subsets of horotori. In [6] the components of \( J_1 \) and \( J_2 \) are produced from ideal points of character varieties and group actions on trees, via the work of Culler and Shalen [14].
In our setting, $J_1$ and $J_2$ will be cusped surfaces produced by Theorem 4.1. Given a pancake $P \subset \mathbb{H}^3$, Theorem 4.1 allows us to assume that $S = J_1$ has an elevation to $\mathbb{H}^3$ that strongly separates $P$. Thus $S$ has a finite cover containing a very large embedded disk $G$, whose elevation $\tilde{G} \subset \mathbb{H}^3$ also separates $P$. By a slight modification of the proof of [6, Theorem 4.2], we may ensure $\Sigma$ lies extremely close to $G$, hence $Q(\Sigma)$ also has an elevation that separates a slightly thicker pancake $\mathcal{N}_\epsilon(P)$. This ensures the ubiquitous condition in Theorem 1.1.

**Theorem 6.1.** Suppose $M$ is a cusped hyperbolic 3–manifold and $\mathcal{S}$ is a finite set of QF surfaces immersed in $M$. Suppose that for each cusp $V$ of $M$, there are two cusps of surfaces in $\mathcal{S}$ that both map into $V$ and have an essential intersection in $V$.

Suppose $P = P(\eta, r)$ is a pancake in $\mathbb{H}^3$, and $\mathcal{S}$ is a surface in $\mathcal{S}$, such that an elevation of $Q(\Sigma)$ to $\mathbb{H}^3$ strongly separates $P$. Then, for any $\epsilon > 0$, there is a closed QF surface $\Sigma$ immersed in $M$ and an elevation of $\text{Core}(\Sigma)$ that strongly separates $\mathcal{N}_\epsilon(P)$. Furthermore, $\Sigma \subset \partial Z$, where $Z$ is a prefabricated manifold whose quasi-Fuchsian pieces are thickenings of covers of surfaces in $\mathcal{S}$.

**Proof.** For most of the proof, we restrict attention to the following special case. Suppose that $\mathcal{S}$ consists of exactly two immersed QF surfaces, $f_1: J_1 \to M$ and $f_2: J_2 \to M$. Suppose, moreover, that for each cusp $V \subset M$, there is a cusp of $J_i$ that maps into $V$ with boundary some multiple of a slope $\alpha_i(V)$, where $\alpha_1(V) \neq \alpha_2(V)$. Finally, suppose that $S = V_1$.

At the end of the proof, we will briefly describe the (purely notational) changes needed to address the general case. For now, we reassure the reader that the special case described above is all that will be needed in the proof of Theorems 1.1 and 1.2.

The maps $f_1: J_1 \to M$ extend to locally isometric immersions $f_1: Q_1 \to M$, where $Q_1 = Q(J_1)$. There is a decomposition of $M$ into a compact set $K$ and a union of horocups $V$, such that for each component $V \subset V$ and for each $i \in \{1, 2\}$, the preimage $f_i^{-1}(V)$ is a non-empty union of vertical rank–1 cusps in $Q_i$. Moreover, each component of $f_i^{-1}(V)$ is an excellent annulus. By a small isotopy, we may arrange that $f_1|_{J_1}$ is transverse to $f_2|_{J_2}$.

Then $D_i = J_i \cap f_i^{-1}(\partial V)$ is a finite set of disjoint, horocyclic simple closed curves. These horocyclic curves cut off the cusps of $J_i$. Moreover $M = f_1(D_1) \cap f_2(D_2) \subset \partial V$ is a finite set and $|f_i^{-1}(x)| \cap \partial D_i| = 1$ for each $x \in M$. The hypotheses imply that that $f_i^{-1}(M)$ contains at least one point in every component of $D_i$. This ensures the ample spiders condition formulated in [6, Definition 3.6, part (W4)], which is crucial for the proof of [6, Theorem 4.2].

From here, we follow the proof of [6, Theorem 4.2], starting at the second paragraph, with $J_i$ consisting of the single connected surface $J_i$. That proof constructs a prefabricated manifold $Z = \mathcal{C} \cup Q_1 \cup Q_2$, where each component of $Q_i$ is a finite cover of $Q_i$, and each component of $\mathcal{C}$ has an elevation that is a finite component of a $V$. That proof ends by verifying the hypotheses of Corollary 5.6, which implies that $Z$ has a convex thickening $Z^+$.

This construction of [6, Theorem 4.2] involves a parameter $\delta > 0$ with the following meaning. For $i \in \{1, 2\}$ the distance in $Z$ between distinct components of $Q_i$ is at least $\delta$. Moreover, the conservative separability Theorem 5.7 ensures that increasing $\delta$ does not change the number of pieces $k + 1 = |\mathcal{C}| + |Q_1| + |Q_2|$ in the construction. (More precisely, this follows from the spider pattern theorem [6, Theorem 2.8].) As a result, one eventually achieves $\delta > 8k$, hence Corollary 5.6 ensures that $Z$ has a convex thickening $Z^+$ with $\partial Z^+$ consisting of closed QF surfaces.

For our purposes, we modify the construction slightly to ensure the pancake condition in the statement of the theorem. This requires increasing $\delta$ even further than what is needed for Corollary 5.6, but again without increasing the number of pieces in $Z$.

Recall that $S = J_1$ has a convex thickening $Q(S)$, with an elevation $\tilde{Q(\Sigma)}$ that strongly separates a pancake $P = P(\eta, r)$. Let $A$ be the rotational axis of $P$, and let $x \in \partial Q(S) \cap A$. Let $x \in \partial Q(S)$ be the projection of $x$. For the value $\epsilon > 0$ in the statement of the theorem, fix $R_{\text{big}} = R(\epsilon) + r$, where $R(\epsilon)$ is as in Theorem 2.4 and $r$ is the radius of the pancake.

Now, we construct a prefabricated manifold $Z$ exactly as above, with the same number of pieces. Recall that $Q_1$ is a disjoint union of finite covers of $Q(S) = Q(J_1)$. By increasing $\delta$, we ensure that in
Theorem 2.4 to \( Z \) playing the role of \( M_0 \), gives

\[
Z^+ := \text{CH}(Z) \subset \mathcal{N}_\epsilon(\hat{Q}; Y) \cup \mathcal{N}_{R(\epsilon)}((Q_1 \setminus \hat{Q}) \cup Q_2 \cup \mathcal{C}; Y).
\]

Since \( \hat{x} \) has distance greater than \( R_{\text{big}} = R(\epsilon) + r \) to any other piece of \( Z \) besides \( \hat{Q} \), it follows that

\[
\mathcal{N}_\epsilon(\hat{x}; Z^+) \subset \mathcal{N}_\epsilon(\hat{Q}; Y) \subset \text{Th}_\epsilon(\hat{Q}).
\]

Let \( \Sigma \) be a component of \( \partial Z^+ \) that passes \( \epsilon \)-close to \( \hat{x} \). Then \( \text{Core}(\Sigma) \subset Z^+ = \text{CH}(Z) \), hence

\[
\text{Core}(\Sigma) \cap \mathcal{N}_\epsilon(\hat{x}) \subset \text{Th}_\epsilon(\hat{Q}).
\]

Now, the projection \( Z^+ \to M \) immerses \( \text{Core}(\Sigma) \) in \( M \), sending \( \hat{x} \) to \( x \). Choose an elevation of \( \text{Core}(\Sigma) \) to \( \mathbb{H}^3 \) so that \( \hat{x} \) lifts to \( \hat{x} \). Recall that \( \hat{x} \) lies on the axis \( A \) of \( P = P(\eta, r) \). By the above equation,

\[
\widehat{\text{Core}}(\Sigma) \cap \mathcal{N}_\epsilon(\hat{x}) \subset \text{Th}_\epsilon(\widehat{Q}(\Sigma)).
\]

Since \( \widehat{Q}(S) \) strongly separates \( P = P(\eta, r) \), it follows that \( \widehat{\text{Core}}(\Sigma) \) strongly separates \( \mathcal{N}_\epsilon(P) \).

Finally, we discuss how to prove the theorem in the general case. The proof is exactly the same, except that the connected surfaces \( J_1 \) and \( J_2 \) are replaced by \( J_1 \) and \( J_2 \), where each \( J_i \) is a (separate) copy of the the finite set of immersed surfaces \( S \). Then \( J_i \) satisfies the ample spiders condition, and can be inserted into the proof of [6, Theorem 4.2] exactly as above. In fact, the proof of [6, Theorem 4.2] is already adapted to finite collections of surfaces, and contains all the necessary book-keeping notation for keeping track of components of \( J_i \). At the end of the construction of [6, Theorem 4.2], one needs to define \( R_{\text{big}} \) exactly as above, and argue in the same way that a surface \( \Sigma \subset \partial Z^+ \) lies very close to \( S \) on a disk of big radius. \( \square \)

**Proof of Theorem 1.1.** By the Kahn–Markovic theorem [24], it suffices to treat the case where \( M \) has cusps. Let \( \Pi, \Pi' \subset \mathbb{H}^3 \) be a pair of planes whose distance is \( 4\eta \). Let \( P^+ = P(2\eta, r) = \mathcal{N}_{2\eta}(D_r) \) be a pancake as in Lemma 3.8. Let \( P = P(\eta, r) = \mathcal{N}_\eta(D_r) \) be a thinner pancake, such that \( P^+ = \mathcal{N}_\eta(P) \).

Now, let \( M = \mathbb{H}^3/\Gamma \) be a cusped hyperbolic 3–manifold. In each cusp \( V \subset M \), choose distinct slopes \( \alpha_1(V) \) and \( \alpha_2(V) \). Then Theorem 4.1 produces a pair of QF surfaces \( J_1 \) and \( J_2 \), such that each \( J_i \) has an elevation that strongly separates \( P \), and such that in every cusp \( V \subset M \), the surface \( J_i \) has a cusp of slope some multiple of \( \alpha_i(V) \). Now, apply Theorem 6.1 with the (connected) surfaces \( J_1 \) and \( J_2 \) and with \( \epsilon = \eta \). That theorem produces a closed QF surface \( \Sigma \) such that an elevation of \( \text{Core}(\Sigma) \) strongly separates \( P^+ = \mathcal{N}_\eta(P) \). By Lemma 3.8, this elevation also separates \( \Pi \) from \( \Pi' \). \( \square \)

7. Ubiquitous surfaces with prescribed immersed slope

In this section, we prove Theorem 1.2, producing a ubiquitous collection of surfaces with prescribed immersed slope. The argument proceeds in two stages. We begin by proving Theorem 7.3, which says that given a slope \( \alpha \) in some cusp of \( M \), there is a QF surface \( F \) immersed in \( M \) and an integer \( m > 0 \) such that all the cusps of \( F \) have slope \( m \cdot \alpha \). Given that Theorem 7.3 only asserts the existence of one surface, without any claim of ubiquity, it is perhaps surprising that this result is new. We wonder whether Theorem 7.3 can be proved without appealing to the Kahn–Markovic theorem [24] using the gluing idea behind prefabricated manifolds [6].

Next, we prove Theorem 1.2 by another analogue of the argument of Theorem 6.1. We take the single QF surface \( F \) produced by Theorem 7.3 and the ubiquitous collection of closed QF surfaces produced by Theorem 1.1. Then, we apply the convex combination theorem to glue together a large regular cover of \( F \) with some large cover of a closed surface splitting a desired pancake.
7.1. One surface with prescribed immersed slope. The idea of of Theorem 7.3 is to build a variant of a prefabricated manifold, whose boundary has cusps. This entails a generalization of prefabricated manifolds.

Definition 7.1. A modified prefabricated manifold is a connected, metrically complete, finite-volume, hyperbolic 3–manifold

\[ Z = \mathcal{C} \cup Q_1 \cup Q_2. \]

Each component of \( Q_i \) and of \( \mathcal{C} \) is a convex hyperbolic 3–manifold called a piece. Each component of \( Q_i \) is a QF 3–manifold with at least one cusp. Each component of \( \mathcal{C} \) is a horocusp. There is a component \( Q \subset Q_1 \) called the special component. One or more rank–1 cusps of \( Q \) called the special cusps. These pieces satisfy the following conditions for \( i \in \{1, 2\} \), and for each component \( C \) of \( \mathcal{C} \):

- (M1) \( Q_i \cap \mathcal{C} \) is the disjoint union of all the cusps in \( Q_i \) that are not special,
- (M2) \( Q_i \cap \partial C \) is an annulus with core curve \( \alpha_i(C) \),
- (M3) \( \alpha_1(C) \) intersects \( \alpha_2(C) \) once transversely,
- (M4) Each component of \( Q_1 \cap Q_2 \) intersects \( \mathcal{C} \).

Thus each component of \( \mathcal{C} \) contains one cusps of \( Q_1 \) and one cusps of \( Q_2 \). In addition to a number of rank–2 cusps (the components of \( \mathcal{C} \)), \( Z \) has some rank–1 cusps, namely the special cusps of \( Q \). The next result is a minor modifications of [6, Proposition 1.6].

Proposition 7.2. Let \( Z = Q_1 \cup Q_2 \cup \mathcal{C} \) be a modified prefabricated manifold with a convex thickening. Then \( \partial Z \neq \emptyset \), and each component of \( \partial Z \) is an incompressible surface. Moreover, every loop in \( \partial Z \) with parabolic holonomy is homotopic into a special cusp.

Proof. Let \( F \) be a surface with non-empty boundary and \( \chi(F) < 0 \). Following [4, Section 7], a (generalized) tubed surface is a 2–complex formed by gluing each component of \( \partial F \) to an essential simple closed curve in a torus. Distinct components of \( \partial F \) are glued to distinct tori.

Given \( Z = \mathcal{C} \cup Q_1 \cup Q_2 \), as in Definition 7.1, set \( Q_1^+ = Q_1 \cup \mathcal{C} \) and \( Q_2^+ = Q_2 \cup \mathcal{C} \). Then each component of \( Q_i^+ \) is a geometrically finite manifold that retracts to a tubed surface. Thus \( Z \) is the union of convex submanifolds, each of which retracts to a tubed surface. The proof that \( \partial Z \) is non-empty and incompressible is now the same as that of [6, Proposition 1.6].

Suppose that a loop \( \gamma \subset \partial Z \) has parabolic holonomy. Since \( Z \) has a convex thickening, \( \gamma \) must be homotopic into some cusp of \( Z \). Suppose, for a contradiction, that \( \gamma \) is homotopic to a loop \( \beta \subset \partial C \) for a cusp \( C \subset \mathcal{C} \). By property (M2), the intersection \( Q_i \cap \partial C \) is an annulus, and by property (M3) the core curves \( \alpha_1(C) \) and \( \alpha_2(C) \) of these annuli have intersection number 1. It follows that \( \beta \) has intersection number \( n \neq 0 \) with at least one of \( \alpha_1(C) \) and \( \alpha_2(C) \). Furthermore, \( n \) depends only on the homology class \( [\beta] = [\gamma] \in H_1(Z) \). Since \( \gamma \) is disjoint from the mid-surfaces of \( Q_1 \) and \( Q_2 \), it follows that \( n = 0 \), which contradicts the hypothesis that \( C \subset \mathcal{C} \).

Since \( \gamma \subset \partial Z \) cannot be homotopic into \( \mathcal{C} \), it must be homotopic into a special cusp. □

Theorem 7.3. Suppose \( M = \mathbb{H}^3/\Gamma \) is a cusped hyperbolic 3–manifold and \( \alpha \) is a slope on \( \partial M \). Then there is a cusped quasi-Fuchsian surface immersed in \( M \) with immersed slope \( \alpha \).

Although the proof of Theorem 7.3 is somewhat technical, the idea is rather simple. Here are the main steps.

1. Construct a prefabricated manifold \( Z = \mathcal{C} \cup Q_1 \cup Q_2 \), with a convex thickening and a local isometry into \( M \). We arrange things so that \( Q_1 \) contains a piece \( Q \) with a cusp mapping to \( m\alpha \).

2. Modify some of the pieces of \( Z \). We replace \( Q \subset Q_1 \) by a 3–fold cyclic cover \( \tilde{Q} \). We also replace some of the rank–2 cusps in \( \mathcal{C} \) by cyclic 3–fold covers. The result is a modified prefabricated manifold \( Z' = \mathcal{C}' \cup Q'_1 \cup Q_2 \), whose special cusps map to \( m\alpha \).

3. Pick a cusped component \( F \subset \partial Z' \), and surger it until has the right properties. Proposition 7.2 gives a way to remove accidental parabolics by surgery, and Theorem 2.2 ensures the resulting surface is quasi-Fuchsian.
Proof of Theorem 7.3. By Theorem 4.1, there is an immersed QF surface $f_1: J_1 \to M$, whose cusps are mapped to some multiple of $\alpha$, as well as at least one slope on every cusp of $M$. Similarly, there is an immersed QF surface $f_2: J_2 \to M$, such that a cusp of $f_1(J_1)$ has distinct slope from a cusp of $f_2(J_2)$ on every horotorus of $M$. We require $f_2(J_2)$ to have nontrivial intersection with $\alpha$.

By Theorem 6.1, there is a prefabricated manifold $Z = Q_1 \cup Q_2 \cup C$, with a convex thickening and a local isometry into $M$. The quasi-Fuchsian pieces of $Q_i$ are thickenings of covers of $J_i$. (In fact, the existence of $Z$ follows from the original, unmodified proof of [6, Theorem 4.2].) By [6, Theorem 2.16], we may build $Z$ to have the following properties. Every quasi-Fuchsian piece $Q_i \subset Q_i$ has an even number of cusps, with $|\partial Q_i| \geq 4$. In addition, the spiders meeting $Q_i$ do not disconnect $Q_i$.

Next, we will modify the pieces of $Z = Q_1 \cup Q_2 \cup C$ to create a modified prefabricated manifold $Z' = Q_1' \cup Q_2' \cup C'$. By construction, $Z$ contains a quasi-Fuchsian piece $Q \subset Q_1$, with a cusp $W \subset Q$ mapping to $m \cdot \alpha$ for some $m \neq 0$. (This manifold $Q$ is a convex thickening of some cover of $J_1$.) There is a connected cyclic 3-fold cover $p: \tilde{Q} \to Q$ with the following properties. First, the cusp $W \subset Q$ has a disconnected preimage $p^{-1}(W) = \tilde{W} \sqcup \tilde{W}' \sqcup \tilde{W}''$ consisting of three isometric lifts of $W$. Every other cusp $V \subset (Q \setminus W)$ has a connected preimage $\tilde{V} = p^{-1}(V)$ that is a 3-fold cyclic cover of $V$. In addition, each component $X \subset Q \cap Q_2$ lifts to $\tilde{Q}$. (By Definition 5.5, each component $X \subset Q \cap Q_2$ is an ideal spider.) The cover $\tilde{Q}$ can be constructed by cutting the mid-surface $S$ of $Q$ along some carefully chosen arcs that avoid the spiders, taking three copies of the cut-up surface, and reassembling. Since $|\partial Q| \geq 4$ and is even, the existence of the arcs with the desired properties and the cover $\tilde{Q} \to Q$ is explained in the proof of Case 2 of [6, Theorem 2.16]. See [6, Page 1215].

For each spider $X \subset Q$, we choose one lift $\tilde{X} \subset \tilde{Q}$. Since each cusp of $Q_1$ contains exactly one spider leg by Definition 5.5, it follows that exactly one of $W, \tilde{W}'$ and $\tilde{W}''$ contains a leg of the chosen lift of some spider. Label the lifts of $W$ so that $\tilde{W}$ is the component that contains a spider leg.

We can now describe the QF pieces of $Z'$, as well as how to glue them. Define $Q_1' = (Q_1 \setminus Q) \cup \tilde{Q}$. Every component $X \subset Q_1 \cap Q_2$ is a spider that corresponds to an isometric spider $\tilde{X} \subset Q_1'$. If $X \subset Q$, the lift $\tilde{X} \subset \tilde{Q}$ was chosen in the previous paragraph. Otherwise, a spider $X \subset (Q_1 \setminus Q)$ corresponds to itself (but is also labeled $\tilde{X}$). Now glue $Q_1'$ to $Q_2$ by isometrically identifying each spider $X \subset Q_2$ with the corresponding spider $\tilde{X} \subset Q_1'$ to obtain a manifold $Z'$.

It remains to construct a union of cusps $C'$, and to glue these rank–2 cusps onto $Z'$ to obtain $Z''$. Every leg of every spider $\tilde{X} \subset Q_1' \cap Q_2$ will run into exactly one rank–2 cusp $\tilde{C} \subset C'$, with distinct legs terminating on distinct cusps. Furthermore, every component $\tilde{C} \subset C'$ will correspond to a rank–2 cusp $C \subset C$, and is a 1-fold or 3-fold cover of $C$. There are three cases, as follows.

First, if $X \subset (Q_1 \setminus Q) \cap Q_2$, then every leg of $X$ lands on a cusp $C \subset C$ disjoint from $Q$. For every such cusp $C$ meeting a leg of $X$, we take an isometric copy $\tilde{C}$ and glue it onto the corresponding leg of $\tilde{X}$. Second, if $X \subset Q \cap Q_2$ has a leg on $W$, there is a unique rank–2 cusp $C_W \supset W$. Construct an isometric lift $\tilde{C}_W$, and embed $\tilde{W}$ into it. By construction, the above-chosen lift $\tilde{X}$ has a leg on $\tilde{W} \subset \tilde{C}_W$. This specifies a way to glue $\tilde{C}_W$ onto $Z'' = Q_1' \cup Q_2$. Third, let $C_Q$ be the union of all other cusps of $\tilde{C}$ (which meet $Q$ but are disjoint from $W$). For every $C \subset C_Q$, Definition 5.5 says that $\partial C$ contains two simple closed curves $\alpha_1(C)$ and $\alpha_2(C)$ that are the cores of $Q_1 \cap \partial C$. These curves form a basis for $\pi_1(C)$. Construct a connected 3-fold cyclic cover $\tilde{C} \to C$ corresponding to the subgroup $\langle 3\alpha_1(C), \alpha_2(C) \rangle \subset \pi_1(C)$. Then we may glue $\tilde{C}$ onto $\tilde{Q} \subset Q_1'$ along a neighborhood of $3\alpha_1(C)$ and onto $Q_2$ along a neighborhood of $\alpha_2(C)$.

We have now constructed a modified prefabricated manifold $Z'' = C' \cup Q_1' \cup Q_2$. The special component is $\tilde{Q}$ and the special cusps are $\tilde{W}'$ and $\tilde{W}''$. Since $Z$ has a convex thickening, then so does $Z'$. In fact, one choice of convex thickening of $Z'$ is obtained by doing the corresponding modifications to the thickened pieces of $Z$. We rename $Z'$ to be this convex thickening. The local isometries $Q_i \to M$ and $C \to M$ define local isometries
\(Q_1' \to M\) and \(Q_2 \to M\) and \(C' \to M\), which agree on the overlaps. Thus \(Z'\) has a local isometry into \(M\). Note that the special cusps \(W'\) and \(W''\) both map to \(m\).

Let \(F\) be a component of \(\partial Z'\) that contains a (rank–1) cusp. By Proposition 7.2, \(F\) is incompressible and every parabolic in \(F\) is homotopic to a special slope. Any accidental parabolics in \(F\) can be removed by surgery, as follows. If there is an accidental parabolic in \(F\), then Jaco’s theorem [23, Theorem VIII.13] provides an annulus \(A\) embedded in \(Z\), with one end an essential simple non-peripheral loop \(\gamma \subset F\) and the other end is a simple closed curve that is a special slope in a special cusp. Surgering \(F\) using \(A\) produces a new surface \(F'\) with \(\chi(F') < 0\), isotopic to a subsurface \(F\), and with fewer accidental parabolics. After a finite number of steps one obtains from \(F\) a surface \(E\) without accidental parabolics that is (isotopic to) a subsurface of \(F\) and with at least one cusp. By Proposition 7.2, the cusps of \(E\) all project to the slope \(m \cdot \alpha\) in \(M\).

By Lemma 2.1, the local isometry \(f: Z' \to M\) extends to a covering map \(f: Y \to M\), where \(\text{Core}(Y) \subset Z' \subset Y\). Since \(Z'\) has finite volume and \(\partial Z' \neq \emptyset\), it follows that \(Y\) is geometrically finite and \(\text{Core}(Y) \neq Y\). Thus, since \(E\) has no accidental parabolics, Theorem 2.2 implies it is quasi-Fuchsian. □

### 7.2. Intersecting and gluing ubiquitous surfaces

In the remainder of this section, we employ another cut-and-paste construction to derive Theorem 1.2 from Theorem 7.3. Before proceeding, we note that only Theorem 1.1 and Theorem 7.3, both of which have already been established, will be needed to derive the cubulation statement of Corollary 1.3 in the next section.

The proof of Theorem 1.2 requires a topological lemma that ensures our surface will not have accidental parabolics.

**Lemma 7.4.** Suppose \(F\) and \(G\) are compact, connected, orientable surfaces with negative Euler characteristic and non-empty boundary. In addition, suppose \(Y = P \cup Q\) is a compact 3–manifold, where \(P = F \times I\) and \(Q = G \times I\). Suppose \(P\) and \(Q\) intersect along a collection of annuli \(A = P \cap Q = \partial G \times I \subset \text{int}(F) \times I\). Let \(\Sigma\) be a component of \(\partial Y \setminus \text{int}(\partial F \times I)\). Suppose every component \(X \subset \Sigma \setminus A\) has \(\chi(X) < 0\). Then \(\Sigma\) is incompressible in \(Y\). If \(\gamma \subset \Sigma\) is a loop that is freely homotopic into \(\partial \Sigma\) through \(Y\), then \(\gamma\) is freely homotopic into \(\partial \Sigma\) through \(\Sigma\).

**Proof.** This is a standard topological argument in the same spirit as Claim 4.4. One needs to consider the intersections between \(A\) and an annulus \(B \subset Y\) realizing a homotopy of \(\gamma\) into \(\partial \Sigma\). The hypothesis \(\chi(X) < 0\) for every component \(X \subset \Sigma \setminus A\) ensures that intersections \(B \cap A\) must indeed occur. □

We also need a geometric lemma about convex hulls. If \(Q\) and \(N\) are convex subsets in \(\mathbb{R}^3\), then \(Q \cap N\) is convex, but \(\partial Q\) can undulate in and out of \(N\), so that \(Q \cap \partial Q\) might be an arbitrary planar surface. For example, consider how a concentric ball and cube might intersect. The same phenomenon happens if \(Q\) and \(N\) are convex submanifolds of a hyperbolic 3–manifold. In particular, \(N \cap \partial Q\) might be compressible in \(\partial Q\). The situation is better when \(Q\) is equal to its convex core.

**Lemma 7.5.** Suppose that \(Q\) and \(N\) are convex 3–dimensional submanifolds of a complete hyperbolic 3–manifold \(Y\), where \(Q = \text{Core}(Q)\) and \(N\) is compact. If \(X\) is a component of \(\partial Q \setminus \text{int}(N)\), then \(\chi(X) < 0\).

**Proof.** The hypothesis \(Q = \text{Core}(Q)\) can be rephrased to say that \(\tilde{Q}\) is the convex hull of its limit set \(\Lambda(\tilde{Q})\). By Carathéodory’s theorem [31, Proposition 5.2.3], every point of \(\tilde{Q}\) is contained in an ideal simplex with vertices in \(\Lambda(\tilde{Q})\). It follows that each \(x \in \partial \tilde{Q}\) is contained in an ideal triangle \(\Delta \subset \partial \tilde{Q}\). Thus \(\partial Q\) cannot have any \(S^2\) or \(T^2\) components, because an elevation of such a component to \(\tilde{Q}\) cannot contain an ideal triangle. In particular, this implies \(X\) is not a sphere or torus.

If \(X\) is a disk or an annulus, we claim that \(X\) is contained in the convex hull of \(\partial X\). Since \(N\) is convex and \(\partial X \subset N\), it follows that \(X \subset N\), which is a contradiction.

Let \(i_1: \pi_1 X \to \pi_1 Q\) be the inclusion–induced homomorphism. If \(i_1(\pi_1 X) = \{1\}\), then \(X\) has an isometric lift \(\tilde{X}\) in \(\partial \tilde{Q}\). Each point \(x \in \tilde{X}\) lies in some ideal triangle \(\Delta \subset \partial \tilde{Q}\). Since \(\tilde{X} = X\) is
compact, each geodesic ray from $x$ to an ideal vertex of $\Delta$ must cross a component of $\partial \tilde{X}$, so $x$ is in the convex hull of these crossing points in $\partial \tilde{X}$. This proves the claim when $i_*(\pi_1 X) = \{1\}$.

If $i_*(\pi_1 X)$ is generated by a parabolic, then a component $\gamma$ of $\partial X \subset N$ is homotopic to a horicycle. But then the convex hull of $\gamma \subset N$ is non-compact, contradicting the hypothesis that $N$ is convex and compact.

In the remaining case, $i_*(\pi_1 X) \cong \mathbb{Z}$ and the core curve of $X$ is homotopic to a closed geodesic $\gamma \subset Q$. Then $X$ has an elevation $\tilde{X} \subset \tilde{Q}$, which lies within a bounded neighborhood of a geodesic $\tilde{\gamma} \subset \mathbb{H}^3$ covering $\gamma$. Each geodesic ray from $x \in \tilde{X}$ to a vertex of $\Delta$ either crosses $\partial \tilde{X}$, or else limits on an endpoint $z$ of $\tilde{\gamma}$. Since $z$ is in the limit set of $\partial \tilde{X}$, it follows that $X$ is contained in the convex hull of $\partial X$. This proves the claim and the lemma. □

**Remark 7.6.** A version of Lemma 7.5 also holds if $Q \subset Y$ is a totally geodesic surface (hence $Q = \text{Core}(Q)$) and $N \subset Y$ is a compact, convex submanifold of any dimension. Under these hypotheses, any component $X$ of $Q \setminus N$ has $\chi(X) < 0$. The proof is easier than Lemma 7.5, because a totally geodesic annulus always lies in the convex hull of its boundary.

A map $f : A \to B$ is **conjugacy injective** if the induced homomorphism $f_*$ sends distinct conjugacy classes in $\pi_1 A$ to distinct conjugacy classes in $\pi_1 B$. The inclusion of a subsurface $A$ into a surface $B$ is conjugacy injective if and only if $\chi(X) < 0$ for any non-peripheral component $X \subset B \setminus A$.

A pair of QF manifolds $Q_1$ and $Q_2$ that are isometrically immersed in a complete hyperbolic 3-manifold $M$ have an **essential intersection** if there are elevations $\tilde{Q}_1, \tilde{Q}_2 \subset \mathbb{H}^3$ such that the limit set $\Lambda(\tilde{Q}_1)$ intersects both components of $S^2_\infty \setminus \Lambda(\tilde{Q}_2)$. It is easy to see this condition is symmetric in $Q_1$ and $Q_2$, and is equivalent to the statement that both components of $\mathbb{H}^3 \setminus \tilde{Q}_2$ contains points in $Q_1$ at unbounded distance from $\tilde{Q}_2$. The next result basically says one can glue together large covers of two QF surfaces to obtain a geometrically finite manifold with QF boundary components that consist of large parts of covers of the two surfaces.

**Proposition 7.7.** For $i \in \{1, 2\}$, suppose that $f_i : J_i \to M$ are immersed QF surfaces in a complete hyperbolic 3–manifold $M$, where one of the $J_i$ is compact. Suppose that $Q_1 = Q(J_1)$ and $Q_2 = Q(J_2)$ have an essential intersection in $M$.

Fix $K > 0$. Then there is a hyperbolic 3-manifold $\tilde{Y} = \tilde{Q}_1 \cup \tilde{Q}_2 = Q(J_1) \cup Q(J_2)$, where $J_i$ is a conservative cover of $J_i$, with an isometric immersion $g : \tilde{Y} \to M$, such that $g|J_i$ is $f_i$ composed with the conservative covering. Moreover, $\tilde{Y}$ has a convex thickening such that $N_K(\tilde{Q}_1 \cap \tilde{Q}_2; \text{Th}_\infty(\tilde{Y})) = \text{Th}_K(\tilde{Q}_1 \cap \tilde{Q}_2)$. Finally, every component of $\partial \tilde{Y}$ is quasi-Fuchsian.

**Proof.** The hypotheses of the proposition are symmetric in $Q_1$ and $Q_2$, hence every statement below applies with the roles of $Q_1$ and $Q_2$ interchanged. For most of the proof, we will work under the additional hypothesis that each $J_i$ is not Fuchsian, hence $Q_i = \text{Core}(J_i) = \text{Core}(Q_i)$. The Fuchsian case is explained at the end.

Fix elevations $\tilde{Q}_i \subset \mathbb{H}^3$ whose limit sets intersect as required, and define $\tilde{N} = \tilde{Q}_1 \cap \tilde{Q}_2$. The virtual convex combination theorem [4, Theorem 5.4] implies that, after replacing each $Q_i$ by a certain finite cover, there is a connected, geometrically finite hyperbolic 3-manifold $\tilde{Y} = Q_1 \cup Q_2$ and a local isometric immersion $g : Y \to M$ such that $g|Q_i = f_i$, and moreover $Y$ has a convex thickening. Since one of the $Q_i$ is compact, it follows that $N = Q_1 \cap Q_2$ is compact and convex. The essential intersection hypothesis implies that $Q_1$ meets both boundary components of $Q_2$.

The subgroup $\pi_1 N$ is finitely generated (because $N$ is compact), free (because it is an infinite index subgroup of a surface group), and contains no parabolics (because $N$ is compact and convex). Since $N$ is irreducible, it must be a handlebody.

**Claim 7.8.** There are conservative covers $\tilde{Q}_1 \to Q_1$, such that

(Y1) $\tilde{N} = \tilde{Q}_1 \cap \tilde{Q}_2$ is an isometric lift of $N$,
(Y2) $\tilde{N}$ does not separate $\tilde{Q}_1$. 
Let enings of the covers $\hat{P}(Y_i)$ also follows by Theorem 5.7 by choosing sufficiently large covers $\hat{Q}_i$.

A thickening of $Th(Y_i)$ is conjugacy injective in $\hat{P}(Y_i)$ cannot be peripheral because of $Q_i$ in added onto the boundary components. We now replace $\partial X$ by $\partial X$ with collars added onto the boundary components. We now replace $F_{2}^{r}$ by $F_{2}^{r}$, since $F_{2}^{r}$ is conjugacy injective in $\partial F_{2}^{r}$, it follows that $\hat{N}$ is conjugacy injective in $Q_i$. The same argument, interchanging the roles of $Q_1$ and $Q_2$, shows that $\hat{N}$ is conjugacy injective in $\hat{Q}_2$.

For notational simplicity, we now replace $\hat{Q}_2$ by $\hat{Q}_2$ and $\hat{N}$ by $\hat{N}^{+}$ and $\hat{Y}$ by $\hat{Y}^{+}$. Since $\hat{N}$ is a handlebody, it is irreducible, and it follows from the $h$–cobordism theorem for 3–manifolds that there is a homeomorphism $h:\partial F \times [-1,1] \rightarrow \hat{N}$ with $h(F \times \{1\}) = \hat{F}_{2}$. The vertical boundary of $\hat{N}$ is $\partial_{v,\hat{N}} = h(\partial F \times [-1,1])$. Observe that $\partial_{v,\hat{N}} \subset \partial \hat{Q}_2 \cup \partial \hat{Q}_2$.

If $\partial_{v,\hat{N}}$ is not entirely contained in $\partial \hat{Q}_2$, then $\partial_{v,\hat{N}} \cap \partial \hat{Q}_2 \neq \emptyset$ and we can isotop $\partial \hat{Q}_2$, shrinking $\hat{Q}_2$, and keeping $\hat{Q}_2$ convex, until $\partial_{v,\hat{N}} \subset \partial \hat{Q}_2$. Let $X$ be a component of $\partial \hat{Q}_2 \setminus \partial \hat{Q}_2$. Since $\hat{N}$ is conjugacy injective in $\hat{Q}_2$, it follows that either $\chi(X) < 0$ or $X$ is peripheral in $\partial \hat{Q}_2$. But $X$ cannot be peripheral because $\partial X \subset \hat{N}$ cannot be parabolic. Thus $\chi(X) < 0$.

It now follows that $P := \hat{Q}_1 \setminus \partial \hat{Q}_1 \approx [\partial _v\hat{Q}_1 \setminus \partial \hat{Q}_2] \times I$. We apply Lemma 7.4 with $Q = \hat{Q}_2$ and $P$ as above to deduce that $\partial Y_+^{r}$ (hence, also $\partial Y_+^{r}$) is incompressible and contains no accidental parabolics. It follows from Theorem 2.2 that every component of $\partial Y_+^{r}$ is QF.

If some $J_i$ is Fuchsian, we merely need to modify the argument that establishes the conjugacy injectivity of $\hat{N}$ in $\hat{Q}_1$. The argument stays the same up to the construction of $\hat{Q}_1$ in Claim 7.8. If $J_i = \text{Core}(J_i)$ is Fuchsian, then $\hat{N} = \hat{Q}_1 \cap \hat{Q}_2$ deformation retracts onto the compact Fuchsian surface $\hat{J}_1 \cap \hat{Q}_2 = \hat{J}_1 \cap \hat{N}$. By Remark 7.6, every component of $\hat{J}_1 \cap \hat{N}$ has negative Euler characteristic. Thus $\hat{N} \cap \hat{J}_1$ is conjugacy injective in $\hat{J}_1$, implying that $\hat{N}$ is conjugacy injective in $\hat{Q}_1$. The rest of the proof is the same. \hfill \qed

We can now complete the proof of Theorem 1.2.

\textbf{Proof of Theorem 1.2.} Let $M = \mathbb{H}^3/\Gamma$ be a cusped hyperbolic 3–manifold, and let $\alpha$ be a slope on one cusp of $M$. Let $\Pi_-$ and $\Pi_+$ be a pair of planes in $\mathbb{H}^3$ whose distance is $4\eta > 0$. Let $P^+ = P(2\eta, r) = N_{2\eta}(D_r)$ be a thinner pancake, so that $P^+ = N_{\epsilon}(P)$. Let $R = R(\epsilon)$ be the constant of Theorem 2.4.

By Theorem 7.3, there is an immersed QF surface $f_1: \hat{J}_1 \rightarrow M$ with immersed slope $\alpha$. Let $Q_1 = Q(\hat{J}_1)$, the work of Shah [36] and Ratner [32], there is a hyperbolic plane $\Pi' \subset \mathbb{H}^3$ that strongly separates $P$, and has an essential intersection with some elevation $\hat{Q}_1$. We also require that $\hat{Q}_1$ lies very far from the pancake $P$; specifically, $d(P, \hat{Q}_1) > R$. Observe that any small perturbation of $\Pi'$ also separates $P$ and has an essential intersection with the same elevation $\hat{Q}_1$.

By Theorem 1.1, closed QF surfaces are ubiquitous in $M$. Thus there is an immersed closed QF surface $f_2: \hat{J}_2 \rightarrow M$ with an elevation that lies arbitrarily close to $\Pi'$. In particular, $\hat{Q}_2 = Q(\hat{J}_2)$ has an elevation $\hat{Q}_2$ that strongly separates the pancake $P$ and has limit points in both components of $S^2_\infty \setminus \hat{Q}_1$. 

\[(Y3) \hat{Y} = \hat{Q}_1 \cup \hat{Q}_2 \text{ has a convex thickening } Th_\infty(\hat{Y}), \]

\[(Y4) Th_K(N) \text{ embeds into } Th_\infty(\hat{Y}). \]
Apply Proposition 7.7 with $Q_1$ and $Q_2$ as above, and with $K = d(P, \hat{Q}_1) > R$, to obtain a hyperbolic 3-manifold $\hat{Y} = \hat{Q}_1 \cup \hat{Q}_2$ with quasi-Fuchsian boundary. By Theorem 2.4, $\partial \hat{Y}$ lies $c$-close to $\partial \hat{Q}_2$ on the portion of $\hat{Y}$ that is $R$-far from $\hat{Q}_1$. Thus a component $\Sigma \subset \partial \hat{Y}$ has an elevation $\Sigma$ that lies $c$-close to $\partial \hat{Q}_2$ on a region that includes the pancake $P$. Thus $\text{Core}(\Sigma)$ strongly separates $P^+ = N^r_\epsilon(P)$.

Finally, observe that the cusps of $\Sigma$ are cusps of $\partial \hat{Q}_1$. Since the cover of $\hat{Q}_1 \to Q_1$ used to construct $\hat{Y}$ is conservative, all cusps of $\Sigma$ map to the same multiple of $\alpha$. \qed

8. Cubulating the fundamental group

In this section, we explain how Corollary 1.3 follows from the preceding theorems and [7, 22]. We begin by briefly reviewing the terminology associated to cube complexes and the groups that act on them. The references [22, 34, 39] give an excellent and detailed description of this material.

For $0 \leq n < \infty$, an $n$–cube is $[-1,1]^n$. A cube complex is the union of a number of cubes, possibly of different dimensions, glued by isometry along their faces. A cube complex $X$ is called $\text{CAT}(0)$ if it is simply connected, and if the link of every vertex is a flag simplicial complex. By a theorem of Gromov, this combinatorial definition is equivalent to the $\text{CAT}(0)$ inequality for geodesic triangles.

A midcube of an $n$–cube is the $(n−1)$ cube obtained by restricting one coordinate to 0. A hyperplane $H \subset \hat{X}$ is a connected union of midcubes, with the property that $H$ intersects every cube of $\hat{X}$ in a midcube or in the empty set. Hyperplanes in a $\text{CAT}(0)$ cube complex $\hat{X}$ are embedded and two-sided, hence they can be used to inductively cut $\hat{X}$ (and its quotients) into smaller pieces. This endows cube complexes and the groups that act on them with a hierarchical structure. See Wise [39], where this philosophy is extensively fleshed out.

Following Sageev [33], group actions on cube complexes can be constructed in the following way. Suppose that $G = \pi_1(Y)$, where $Y$ is a compact cell complex. Then $G$ acts by deck transformations on $\hat{Y}$, and is quasi-isometric to $\hat{Y}$. Suppose that $H_1, \ldots, H_k$ are codimension–1 subgroups of $G$, meaning that some metric thickening of an orbit $(H_i)\hat{y}$ separates $\hat{Y}$ into non-compact components. Sageev uses this data to build a $G$–action on a dual cube complex $\hat{X}$, whose hyperplanes are in bijective correspondence with cosets of the $H_i$. See Hruska and Wise [22] or Sageev [34] for detailed, self-contained expositions.

In our application, $\hat{Y}$ will be the compact part of a hyperbolic 3–manifold $M$. The codimension–1 subgroups of $G = \pi_1Y = \pi_1M$ will be QF surface groups. The quasi-isometry $G \to \hat{Y}$ identifies the cosets of $\pi_1(S)$ with the elevations of $S$ in $\hat{Y}$. These will give rise to hyperplanes in $\hat{X}$.

The $G$–action on $\hat{X}$ is called proper if the stabilizer of every cube is finite. If $G$ is a torsion-free group, then proper actions are free, i.e. point stabilizers are trivial, hence $G$ acts on $\hat{X}$ by deck transformations. The $G$–action on $\hat{X}$ is called cocompact if the quotient $X = \hat{X}/G$ is compact.

The following is a special case of a theorem of Bergeron and Wise [7, Theorem 5.1].

**Theorem 8.1.** Let $M = \mathbb{H}^3/\Gamma$ be a cusped hyperbolic 3–manifold. Suppose that $M$ contains an ubiquitous collection $\mathcal{S}$ of quasi-Fuchsian surfaces, with the property that for every cusp $V \subset M$, the cusps of surfaces in $\mathcal{S}$ map to at least two distinct immersed slopes in $V$. Then there are finitely many surfaces $S_1, \ldots, S_k \in \mathcal{S}$ such that $\Gamma = \pi_1M$ acts freely on the $\text{CAT}(0)$ cube complex $\hat{X}$ dual to all elevations of $S_1, \ldots, S_k$ to $\mathbb{H}^3$.

Combining Theorem 8.1 with Theorem 1.4 gives

**Corollary 8.2.** Let $M = \mathbb{H}^3/\Gamma$ be a cusped hyperbolic 3–manifold. Let $\mathcal{S}$ be the collection of surfaces guaranteed by Theorem 1.4. Then $\Gamma = \pi_1M$ acts freely on a finite dimensional $\text{CAT}(0)$ cube complex $\hat{X}$, with finitely many $\Gamma$–orbits of hyperplanes. Each (immersed) hyperplane in $X = \hat{X}/\Gamma$ corresponds to an immersed, cusped quasi-Fuchsian surface in $\mathcal{S}$. 
We emphasize that in Corollary 8.2, the quotient \( X = \tilde{X}/\Gamma \) need not be compact. However, this quotient has finitely many immersed hyperplanes. Such a cubulation of \( \Gamma \) is called \textit{co-sparse}. While weaker than the cocompact cubulation described below, a co-sparse cubulation is nonetheless sufficient for many applications such as virtual specialness. See [7, Proposition 3.3].

Stronger hypotheses on \( S \) guarantee that the action on \( \tilde{X} \) is cocompact.

**Theorem 8.3.** Let \( M = \mathbb{H}^3/\Gamma \) be a cusped hyperbolic 3–manifold. Suppose that \( M \) contains an ubiquitous collection \( S \) of quasi-Fuchsian surfaces, with the property that for every cusp \( V \subset M \), the cusps of surfaces in \( S \) map to exactly two distinct immersed slopes \( \alpha(V) \) and \( \beta(V) \). Then there are finitely many surfaces \( S_1, \ldots, S_k \in S \) such that \( \Gamma = \pi_1M \) acts freely and cocompactly on the CAT(0) cube complex \( \tilde{X} \) dual to all elevations of \( S_1, \ldots, S_k \) to \( \mathbb{H}^3 \).

**Proof.** Let \( S_1, \ldots, S_k \) be the finite collection of surfaces guaranteed by Theorem 8.1. Then \( \Gamma \) acts freely on the cube complex \( \tilde{X} \) dual to these surfaces. For each \( S_i \), let \( H_i = \rho(\pi_1S_i) \) be the corresponding subgroup of \( \Gamma \). Let \( V_1, \ldots, V_n \) be the horocusps of \( M \).

Hruska and Wise [22, Theorem 1.1] show that the \( \Gamma \)–action on the dual cube complex \( \tilde{X} \) is \textit{relatively cocompact}. This means that the quotient \( X = \tilde{X}/\Gamma \) decomposes into a compact cube complex \( K \) and cube complexes \( C_1, \ldots, C_n \) corresponding to the cusps \( V_1, \ldots, V_n \). Each \( C_j \) is a quotient under \( \pi_1(V_j) \cong \mathbb{Z}^2 \) of a cube complex \( \tilde{C}_j \subset \tilde{X} \). Furthermore, \( \tilde{C}_j \subset \tilde{X} \) is exactly the dual cube complex constructed from the parabolic subgroups \( \pi_1(S_i) \cap \pi_1(V_j) \) acting on \( \tilde{V}_j \).

In the case at hand, every peripheral group \( \pi_1(V_j) \cong \mathbb{Z}^2 \) intersects the fundamental groups of the \( S_i \) in exactly two parabolic subgroups, namely \( \langle \alpha(V_j) \rangle \) and \( \langle \beta(V_j) \rangle \). The slopes \( \alpha(V_j) \) and \( \beta(V_j) \) are distinct. Thus \( \tilde{C}_j \cong \mathbb{R}^2 \) is the standard cubulation of \( \mathbb{Z}^2 \), constructed from two families of parallel lines in the plane. See [39, Figure 6.4]. In particular, every \( C_j \) is a compact torus, hence \( X = \tilde{X}/\Gamma \) is compact as well. \( \square \)

**Proof of Corollary 1.3.** Let \( M \) be a complete hyperbolic 3–manifold. If \( M \) is closed, then [7, Theorem 1.5] says that \( M \) acts freely and cocompactly on a cube complex \( \tilde{X} \) dual to the elevations of finitely many Kahn–Markovic surfaces.

If \( M \) is non-compact, let \( V_1, \ldots, V_n \) be its cusps. For every \( V_j \), choose two distinct slopes \( \alpha(V_j) \) and \( \beta(V_j) \). For each \( V_j \), let \( S_{\alpha,j} \) and \( S_{\beta,j} \) be two quasi-Fuchsian surfaces produced by Theorem 7.3, whose immersed boundary slopes are \( \alpha(V_j) \) and \( \beta(V_j) \) respectively. Let \( S \) be the (infinite) collection of QF surfaces consisting of \( \{S_{\alpha,1}, \ldots, S_{\alpha,n}, S_{\beta,1}, \ldots, S_{\beta,n}\} \) and the ubiquitous collection of closed QF surfaces guaranteed by Theorem 1.1. In particular, every \( S \in S \) is either closed or has all cusps mapping to exactly one multiple of \( \alpha(V_j) \) or \( \beta(V_j) \), for one \( V_j \).

Now, Theorem 8.3 applied to \( S \) gives a free and cocompact action on a dual cube complex \( \tilde{X} \). \( \square \)

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